SOME PROBLEMS IN BLOW-UP

By

Michael Grant McGuigan

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Abstract

This work considers the blow-up behaviour of the semilinear initial value problems $u_t - \nabla^2 u + \lambda e^u$ (problem (A)) and $u_t - \nabla^2 u + u^p \pm u^\alpha |\nabla u|^\beta$ with $p > 1$ and $\alpha, \beta \geq 0$ (problem (B)).

A preliminary chapter describes the physico-chemical background and summarises known blow-up results for the problems (A) and (B).

Chapter 1 considers the problem (A) with homogeneous Dirichlet boundary conditions and positive initial data. We take λ slightly larger than λ^* the supremum of the closed spectrum of the steady-state problem. By using the method of upper and lower solutions we find T such that the blow-up time, t_b, is bounded from above and below 'close to' $T(\lambda - \lambda^*)^{-1/2}$.

Chapter 2 considers the problem (B) in one space dimension. The main result is that, unless $\pm u^\alpha u_x^\beta - +u^\alpha |u_x|^\beta$, then blow-up is possible for $\alpha \geq 0, \beta > 1$, and $p > \alpha + \beta$. If $\pm u^\alpha u_x^\beta - +u^\alpha |u_x|^\beta$ then blow-up can occur for all $\alpha, \beta \geq 0$. Additionally we find a range of α, β, and p for which, with suitable initial data, the blow-up will occur at a single point. Estimates of the blow-up rate are also obtained.

Chapter 3 considers the higher dimensional problem (B). Results analogous to the one-dimensional case are obtained (with single-point blow-up now occurring for radially symmetric solutions). For non-symmetric solutions we find that blow-up occurs within a compact subset of a convex domain whenever symmetric solutions would blow up at a single point (i.e. for the same α, β and p).

Introduction

The work of this thesis is concerned with the phenomenon of finite-time blow-up as it applies to the solutions of semilinear initial value problems. Two specific equations are considered; in Chapter 1, $u(x,t)$ satisfies $u_t = \nabla^2 u + \lambda e^u$, and in Chapters 2 and 3 $u(x,t)$ is the solution to $u_t = \nabla^2 u + u^p \pm u^\alpha |\nabla u|^\beta$.

In both cases the equations hold within bounded subsets of N-dimensional space (usually N=3 for practical interest) with appropriate boundary conditions.

In the second problem, the form of nonlinearity and gradient terms are chosen because for all $p > 1$ the possibility of finite-time blow-up exists and there is scope for the combative influences of the gradient term to lead to interesting results.

The semilinear initial value problem studied in Chapter 1 was first applied to combustion problems by Frank-Kamenetskii (1939) and has become known to mathematicians through the work of I.M. Gelfland (1963).

Gelfland's motivation for studying the problem was the phenomenon of self-ignition (blow-up).

Since that time, the blow-up behaviour of solutions to this type of equation has received a great deal of interest and a wide range of results in this area are known (see Section P.2).

The aim of this work is to extend this knowledge, and further, to address some of the 'standard' blow-up questions in the case of the less widely studied problems of Chapters 2 and 3.

In Chapter 1, we obtain the first result by establishing an analytical estimate for the blow-up time of the semilinear heat equation with exponential nonlinearity and when λ is slightly larger than λ^* (λ^* is the supremum of the spectrum, which we take to be closed, of the associated steady-state problem $\nabla^2\omega + \lambda e^{\omega} = 0$ in Ω and with corresponding boundary conditions). This analysis serves to ratify an estimate previously obtained in Lacey 1983 by formal asymptotics which found that, if λ is slightly larger than λ^* then the blow-up time is of the order of $(\lambda - \lambda^*)^{-1/2}$.

In Chapters 2 and 3 we consider the semilinear heat equation with a gradient term and with Dirichlet boundary conditions. Starting with the one-dimensional equation in Chapter 2, the questions addressed are generally of the form

(i) for what range of the parameters α, β and p is blow-up possible,

(ii) if blow-up occurs, can anything be said about the blow-up set, and

(iii) if blow-up occurs, can anything be said about the 'rate' at which u approaches the blow-up time.

The higher dimensional problem is considered in Chapter 3.

In most cases, albeit for particular functions, or particular regions, partial answers to each of the questions (i)-(iii) are obtained.

Generally, we find that an important parameter in determining the blow-up behaviour of solutions to this equation is $p - (\alpha + \beta)$. In the case of a negative gradient term we find that, if $p \leq \alpha + \beta$, then finite-time blow-up is impossible.

Also, if $p > \alpha + \beta$, then for suitable initial data finite-time blow-up will occur.

For a positive gradient term, we find that the sign of $p - (\alpha + \beta)$ determines the split between where we are, and where we are not able to establish single-point blow-up for certain cases, e.g. the one-dimensional problem with u having a unique local maximum, and the symmetric case in higher dimensions. In the asymmetric case, this same split is observed between where we are, and where we are not able to establish that the blow-up occurs within a compact subset of a convex domain.

The belief is that this behaviour is a reflection of the nature of the solutions and not simply a failure of the techniques used although it has not been able to very this.

Another parameter which arises naturally in the case of a negative gradient term is $\beta - 2(p-\alpha)/(p+1)$. Generally, maximum principles for u and its derivatives prove difficult to establish because of the presence of the gradient terms.

When $\beta < 2(p-\alpha)/(p+1)$, however, we find that usually a naturally arising gradient term of the desired sign exists which dominates over the presence of terms associated with the $u^\alpha|\nabla u|^\beta$ term.

When $\beta > 2(p-\alpha)/(p+1)$ the dominating gradient term would be unmanageable without a bound for the gradient. Such a bound has been established and its derivation is contained separately in Appendix A.

Preliminaries

P.1 Background

In Chapter 1 we discuss the well-known reaction-diffusion equation or semilinear heat equation. The reaction-diffusion equation arises as a mathematical model of chemically reacting systems both in the absence and presence of diffusion of mass. Thus a branch of physical chemistry, chemical kinetics, had developed which is concerned with the chemical processes involving the rate and mechanisms of reactions in addition to the physical processes of heat and material transport.
Both mass transfer by molecular diffusion and heat transfer by conduction can be described by similar equations and may consequently be considered as analogous. During the course of a reaction, both heat conduction and diffusion of mass may take place. Reactions may be described by the 'rate' of the reaction, i.e. the rate of heat generation or absorption and of the production or consumption of some chemicals. Generally, the reaction rate can be assumed to depend upon concentrations of the reactants as a power law; temperature dependence is given by the Arrhenius law (see below).
The general equations of reaction-diffusion are derived by the principle of continuity to each of the reacting chemicals and to the enthalpy. In the case of reaction-diffusion in a porous catalyst, for example, the reaction converts a gas into useful products when catalysed heterogeneously. The catalyst comes in the form of a porous pellet and the reacting gas has to diffuse into the interior of this if the catalyst there is to be useful.

Hence, if T denotes temperature, and there are s reacting species with concentrations $C_1, C_2, \ldots C_s$ then the course of the reaction is described by a system of equations of heat conduction and mass diffusion of the form

$$\mathscr{E}\frac{\partial C_i}{\partial t} = \sum_{k=1}^{s} \nabla\cdot(D_{ik}\nabla C_k) + \rho S\sum_{j=1}^{R} \alpha_{ji} r_j \quad (i = 1, \ldots s)$$

$$C_p\frac{\partial T}{\partial t} = \nabla\cdot(K\nabla T) + \rho S\sum_{j=1}^{R} (-\Delta H)_j r_j$$

where D_{ik} are the multicomponent diffusion coefficients, ρS represents the catalytic area per unit volume, α_{ji} is the stoichiometric coefficient of the jth reaction, r_j the rate of the jth reaction, C_p the heat capacity, K the thermal conductivity, and $(\Delta H)_j$ the heat of the jth reaction.

For a single reaction, the suffixes may be dropped, and the equations in non-dimensional form become

$$\frac{\partial C}{\partial t} = \nabla\cdot(D\nabla C) - \epsilon\eta R(C, T)$$

$$\frac{\partial T}{\partial t} = \nabla\cdot(K\nabla T) + \eta R(C, T)$$

where R represents the reaction rate and $\epsilon = \frac{C_p\rho}{\Delta H}$.

If the coefficients of heat conduction and diffusion are assumed constant, then

$$\frac{\partial C}{\partial t} = D\nabla^2 C - \epsilon\eta R(C,T)$$

$$\frac{\partial T}{\partial t} = K\nabla^2 T + \eta R(C,T)$$

where ∇^2 is the Laplacian.

If the reaction rate has a power law dependence on the concentration, C, and the Arrhenius temperature dependence, then $R(C,T)$ takes the form

$$R(C,T) \sim C^p \exp\left\{-\frac{E}{RT}\right\}$$

where the constant E is the activation energy, R is the universal gas constant, T is the local temperature, and p is a positive number which denotes the reaction order.

One type of behaviour which may be observed is self-ignition. If the heat of reaction is large, and the heat loss is small, a substantial increase in temperature can occur before there is a significant depletion of the reactant. After the temperature becomes high, there is a consequent reduction in the reactant concentration and the reaction rate slows.

Alternatively, if the rate of reaction is not large, then a shortage of reactant must occur before the temperature becomes high. Again if the heat loss is not small, the temperature must remain low, even without the reduction in reactant.

In the case of the 'zeroth-order' reaction, the rate term is independent of concentration $(p=0)$ and the equation governing the temperature becomes

$$\frac{\partial T}{\partial t} = K\nabla^2 T + \eta \exp[-E/RT].$$

Alternatively, if the heat of the reaction is large (so that the parameter ϵ is small) and the initial reactant concentration, c_0 say, is independent of position, one can, initially at least, neglect the burning up of the combustible component, in which case $c = c_0$.

Again, the temperature will be governed by an equation of the form

$$\frac{\partial T}{\partial t} = K\nabla^2 T + \eta c_0^p \exp[-E/RT].$$

To study the behaviour of this equation, it is usually necessary to employ some approximation to the temperature dependence of the rate constant. When there is a significantly large activation energy and heat of reaction, the most popular approximation as been the Frank-Kamenetskii (positive exponential) approximation for the Arrhenius term $\exp[-E/RT]$:-

$$\exp[-E/RT] = \exp\left[\frac{-E}{RT_a}\right]\exp\left[\frac{u}{1+\delta u}\right] \sim \exp\left[\frac{-E}{RT_a}\right]e^{u}$$

where T_a is the ambient temperature, $\delta = \frac{RT_a}{E}$ will be small, and

$u = \frac{E(T-T_a)}{RT_a^2}$ is a dimensionless temperature excess above ambient.

In this case, the temperature equation reduces to

$$\frac{\partial u}{\partial t} = \nabla^2 u + \lambda e^{u}$$

where λ is a positive dimensionless parameter (the Frank-Kamenetskii parameter), which depends upon, among other things, the size of the vessel. Alternatively, a single power of the non-dimensional temperature excess u, say u^p, may be used, in which case the equations are analogous to those governing pth order isothermal reactions (but with a generation of heat rather than a disappearance of reactant). Similar equations also arise in the context of astrophysics (Fowler 1914 and 1931).

In general terms, therefore, the equation of interest is usually expressed in the form

$$u_t = \nabla^2 u + \lambda f(u)$$

where f is a positive, increasing, convex function of u.

The boundary condition usually attributed to the non-dimensional reaction-diffusion equation is expressed as

$$\frac{\partial u}{\partial n} + \beta(x)u = 0 \quad \text{on the boundary,}$$

where n is the outward unit normal and $0 < \beta(x) \leq \infty$.

At the boundary of the vessel within which the reaction takes place, the situation is in reality extremely complicated. If $\beta(x) = \infty$ at all points on the boundary, then we have Dirichlet boundary conditions, $u = 0$. This corresponds to the situation where the temperature (or concentration) may be legitimately specified at the surface, and it is this type of boundary condition which appears most often. The legitimacy of Dirichlet boundary conditions can be questioned, however, as the surface temperature and concentration will be determined by heat and mass transfer from within the reacting medium. Hence, a possibly less idealistic boundary condition is that expressed above where β is not infinite and giving instead Robin conditions.

Physical justification for including the gradient term in the reaction-diffusion equation to form the problem studied in Chapters 2 and 3 is less well grounded. Here the equation takes the form

$$u_t = \nabla^2 u + u^p \pm u^\alpha |\nabla u|^\beta$$

where $p > 1,\ \alpha, \beta \geq 0$ and the problem has positive initial data and homogeneous Dirichlet boundary conditions.

Without the gradient term, this would correspond to the semilinear heat equation previously described with the Arrhenius temperature dependence approximated by a single power of the temperature excess. With the additional term, the suggestion is that the reaction depends, in some way, on the gradient. The gradient term may also represent some highly nonlinear form of convection. An example of chemical reactions which depend on convection is the tubular reactor (see for example Amondson and Raymond 1964).

However, the equation itself, and this particular form of gradient term are here primarily studied out of mathematical interest as opposed to any belief that this particular equation represents an accurate model of some known chemical reaction.

Throughout this work we assume that Ω is a bounded domain in $\mathbf{R}^N$ with smooth boundary $\partial\Omega$, and where $\frac{\partial\Omega}{\partial n}$ denotes the outward normal derivative to $\partial\Omega$.

Problem (A) is the aforementioned reaction-diffusion initial-boundary value problem

$$u_t = \nabla^2 u + \lambda f(u) \qquad x \text{ in } \Omega, t > 0$$

$$u(x,0) = u_0(x) \qquad x \text{ in } \Omega$$

$$\frac{\partial u}{\partial n} + \beta u = 0 \qquad x \text{ on } \partial\Omega, t > 0$$

where $0 < \beta(x) \leq \infty$, and $u = u(x,t)$. Chapter 1 considers this problem in the case $f(u) = e^u$. The corresponding steady-state problem (Problem (A2)) can be written

$$\nabla^2\omega + \lambda f(\omega) = 0 \qquad x \text{ in } \Omega$$

$$\frac{\partial\omega}{\partial n} + \beta\omega = 0 \qquad x \text{ on } \partial\Omega,$$

where $\omega = \omega(x)$.

In the problem (B) it is assumed that the reaction-diffusion equation also depends on convection through the presence of a (possibly nonlinear) gradient term. The solution is again denoted by $u(x,t)$ and Dirichlet boundary conditions are applied. Hence

$$u_t = \nabla^2 u + f(u) - G(u,\nabla u) \qquad x \text{ in } \Omega, t > 0$$

$$u(x,0) = \varphi(x) \qquad x \text{ in } \Omega,$$

$$u(x,t) = 0 \qquad x \text{ on } \partial\Omega, t > 0.$$

In Chapters 2 and 3 this problem is considered in the case $f(u) = u^p,\ p > 1$ and $G(u,\nabla u) = \pm\, u^\alpha|\nabla u|^\beta$ with $\alpha \geq 0, \beta \geq 1$.

P.2 Known results on blow-up

The statements, u blows up, exhibits thermal runaway or has finite escape time are equivalent and mean that the function u becomes infinite in a finite time.

It is well known that solutions to the problem (A) may cease to exist in a finite time (Fujita 1969, Bebernes & Kassoy 1981, Lacey 1983), although as pointed out by Ball (1977 & 1978) such behaviour need not always be as a result of blow-up. Ball 1978 notes that for ordinary differential equations a standard existence and continuation theorem (Hartman 1964) allows the conclusion that global non-existence and blow-up are equivalent. For infinite dimensional initial value problems such as those arising from partial differential equations, however, a general statement relating blow-up and non-existence would require the existence of an analogous continuation theorem which need not always be the case. Further, such a theorem would only indicate that <u>some</u> norm of u blows up.

Hence, the co-existence of non-equivalent norms each of which can serve as a measure of the size of a solution means that it is possible for a solution to cease to exist in a finite time through losing the appropriate degree of smoothness. In Ball 1978 an example is given where global non-existence of a solution is observed while (through lack of a suitable existence theory) no definite assertion concerning blow-up of this solution is possible.
Tzanetis 1986, however, establishes that for the problem (A), global non-existence and blow-up <u>are</u> equivalent for problems of the type considered here (see also Caffarelli & Friedman 1988).
The existence of blow-up for the solutions to such problems also need not be related to any physical phenomenon. Hence, although the problem (A) can be

taken as a model of a chemical reaction for which the blow-up of solutions may characterise a thermal explosion, blow-up also indicates that the mathematical model becomes invalid near the blow-up time.

If finite-time blow-up does occur then numerous subsequent questions arise, some of the most interesting of which are related to the location, in the spatial domain, of the blow-up points, the precise time at which finite-time blow-up occurs, and the behaviour of u close to the blow-up time and position.

Quoting the definition given in Friedman & MacLeod 1985, if T is the finite blow-up time of a function u then a point x in the spatial domain Ω is called a 'blow-up point' if there exists a sequence (x_m, t_m) such that

$$t_m \nearrow T,\ x_m \to x \quad \text{and} \quad u(x_m, t_m) \to \infty \quad \text{as} \quad m \to \infty.$$

The set of all points x in Ω which are blow-up points is then the blow-up set.

The nature of the blow-up set is one route by which finite-time blow-up may be characterised and has led to the following definitions:-

Single-point blow-up means that the blow-up set is comprised of a single point (see for example Weissler 1983, Friedman & MacLeod 1985 and Mueller & Weissler 1985).

Regional blow-up means that there exists some subregion of Ω, say $\bar{D}$, such that each x in D is a blow-up point, and, Global blow-up means that every x in Ω is a blow-up point, i.e. blow-up occurs throughout Ω.

Examples of regional and global blow-up for the problem (A) can be found in Lacey 1986 and Galaktionov & Posashkov 1988.

The 'form' of blow-up can also be characterised according to the behaviour of u after the blow-up time, and, for example,

Complete blow-up means that for any x in Ω, $u(x,t) = \infty$ for $t > T$, and

Partial blow-up means that the solution u continues to exist in some sense, for at least some small time after T (see for example Baras & Cohen 1987).

A necessary condition for the blow-up of solutions to the problem (A) is that

$$\int_a^\infty \frac{ds}{f(s)} < \infty \tag{C1}$$

for finite a. This can be established (as in Lacey 1983 and Sperb 1981) by noting that the solution, $z(t)$, of the ordinary differential equation

$$\frac{dz}{dt} = \lambda f(z), \quad z(0) = \sup_{x\in\Omega} u_0(x) \tag{D}$$

is an upper solution to u, i.e.

$$z_t \geq \nabla^2 z + \lambda f(z) \text{ in } \Omega, \quad z(0) \geq u_0(x) \text{ and } \frac{\partial z}{\partial n} + \beta z \geq 0 \text{ on } \partial\Omega,$$

so that, by comparison,

$$z(t) \geq u(x,t) \quad \text{for} \quad x \quad \text{in} \quad \Omega,$$

so long as both z and u exist.

This function, $z(t)$, satisfies

$$\lambda t = \int_{z(0)}^{z(t)} \frac{ds}{f(s)}$$

so that, if

$$\infty = \int_{z(0)}^{\infty} \frac{ds}{f(s)}$$

then finite-time blow-up of u would be impossible (as z can tend to infinity only with infinite time).

Clearly, the necessary condition (C1) also applies to the nonlinearity f appearing in the more general problem (B).

This condition (C1) is satisfied by any f which, for large $s,$ grows at least as fast as $s(\log s)^{1+b},\ b > 0$ and hence by the functions e^u and $u^p,\ p > 1$ considered herein.

In Chapter 1, the problem (A) is considered in the particular case of $f(u) = e^u$. It is anticipated, however, that a similar analysis should be possible for any positive, increasing, convex function f which satisfies the condition (C1).

Functions, $\omega(x)$ which solve the problem (A2) are possible steady-states for the corresponding parabolic problem (A) and it is in this context that many nonlinear elliptic problems are studied. Solutions to the problem (A) will have many features in common with both solutions to (A2) and with the ordinary differential equation (D).

It is well known that there exists a critical value of λ , say λ^* , such that for $0 < \lambda < \lambda^*$, the problem (A2) has a classical solution, whereas for $\lambda > \lambda^*$ no solution exists (Keller & Cohen 1967, Amann 1976). The maximum principle (Protter & Weinberger 1967) applies to the problem (A2) for the considered nonlinearities and establishes that solutions to (A2) are positive. If the set of λ for which there exists positive solutions to (A2) is the spectrum of (A2), then λ^* is the supremum of the spectrum. λ^* may or may not belong to the spectrum depending on the particular function f and region Ω , e.g. if $f(u) = e^u$, then for a radially symmetric region λ^* is in the spectrum of (A2) for dimension $1 \leq N \leq 9$ and is not in the spectrum if $N \geq 10$ (Joseph & Lundgren 1973). Crandall & Rabinowitz 1975 give conditions on the behaviour of $f(s)$ for large values of s which ensure that λ^* is in the spectrum for the Dirichlet problem $(u = 0$ on $\partial\Omega)$.

For positive, increasing, convex f satisfying (C1) then if $\lambda > \lambda^*$ there is finite-time blow-up of $u,$ the solution to problem (A). (Fujita 1969, Pao 1977 & 1978, Bebernes & Kassoy 1981, Lacey 1983 and Bellout 1987).

Even if λ is small enough for there to be a steady-state solution to (A2), it is still possible for there to be blow-up if the initial data, $u_0(x),$ is sufficiently large. Indeed, if there are at least two steady-state solutions, say $\omega_m < \omega,$ then u blows up if $u_0 > \omega$ (Fujita 1969, Lacey 1983).

For solutions to the problem (A) which do blow-up there are also a number of results concerning the blow-up set.

For a radially symmetric problem, say Ω the unit ball with $u_0 = u_0(r)$ where r is the distance from the centre and u_0 decreasing in $r,$ or for a one-dimensional problem with u_0 having a unique local maximum, then solutions to the problem (A) will blow up at a single point provided f grows fast enough (Friedman & MacLeod 1985, see also Weissler 1984 and Mueller & Weissler 1985).

For the radially symmetric problems the unique blow-up point is the origin.

The growth conditions on the function f to give single-point blow-up require that there exists some positive $F(s)$ such that $\int_a^\infty \frac{ds}{F(s)} < \infty$,

$$\text{and } f'(s)F(s) - f(s)F'(s) \geq \epsilon F(s)F'(s) \text{ for some } \epsilon > 0. \tag{C2}$$

This condition is satisfied if f grows exponentially, as a power, or as fast as

$s(\log s)^{1+b}$ for $b > 1$.

For $f(s) \sim s(\log s)^{1+b}$ we recall that $b > 0$ is required to allow the possibility of blow-up, i.e. to satisfy the necessary condition (C1). The condition (C2) for single-point blow-up is not, however, satisfied if f only grows as fast as

$s(\log s)^{1+b}$ for $0 < b \leq 1$.

Lacey 1986 establishes complementary results to those of Friedman & MacLeod 1985 by showing that for these 'slower growing' nonlinearities, single-point blow-up does not occur. For $f(s) \sim s(\log s)^{1+b}$ for large values of s, then Lacey 1986 shows that for $0 < b \leq 1$ there is global blow-up and for $b = 1$ there is regional blow-up which will be global for certain regions Ω. If the problem (A) has Dirichlet boundary conditions ($u = 0$ on $\partial\Omega$) then Friedman & MacLeod 1985 also find that the growth condition (C2) ensures that blow-up will occur within a compact subset of a convex domain Ω.

The specific problem addressed in Chapter 1 is concerned with identifying the finite-time, say T, at which blow-up occurs.

If $f(u) = e^u$ in problem (A) and if $\lambda > \lambda^*$ where λ^* is in the spectrum of the steady-state problem (A2), then Lacey 1983 shows that there is a time $t = t_u(\lambda - \lambda^*)^{-\frac{1}{2}}$ by which the solution u must have blown up.

If the steady-state solution for $\lambda = \lambda^*$ is denoted by ω^*, then Lacey 1983 also finds that, if

$$u_0(x) < \omega^*(x) \quad \text{in} \quad \Omega,$$
$$\text{with} \qquad \frac{\partial u_0}{\partial n} > \frac{\partial \omega^*}{\partial n} \quad \text{or} \quad u_0 < \omega^* \quad \text{on} \quad \partial\Omega \tag{C3}$$

then there also exists a time, $t = t_l(\lambda - \lambda^*)^{-\frac{1}{2}}$ at which the solution, u, still exists.

In this case, it follows that

$$t_l(\lambda - \lambda^*)^{-\frac{1}{2}} < T < t_u(\lambda - \lambda^*)^{-\frac{1}{2}}$$

(see also Chapter 1).

Lacey 1984 shows that, for 'large' values of λ, then

$$T = O(1/\lambda).$$

In Chapters 2 and 3 we are also interested in determining bounds for the rate at which a solution to the problem (B) can blow up at the time T. In this case, $f(u) = u^p,\ p > 1$ in (B). These types of results exist for the problem (A) and with this nonlinearity it can be established (as in Friedman & MacLeod 1985) that

$$\max_{x\in\Omega} u(x,t) \geq \frac{C}{(T-t)^{1/(p-1)}}$$

for $0 < t < T$ and some $C > 0$.

The complementary inequality, i.e.

$$u(x,t)) \leq \frac{C}{(T-t)^{1/(p-1)}} \tag{I1}$$

is, however, more difficult to come by. The inequality (I1) asserts that u blows up at the same rate as the solution of the ordinary differential equation (D) (with $f(s) = s^p,\ p > 1$) and was first proved by Weissler in 1985 for radial solutions with rather special initial data.

Friedman & MacLeod 1985 subsequently proved (I1) provided Ω is a bounded convex domain and the initial data u_0 satisfies

$$u_0 \geq 0 \quad \text{and} \quad \nabla^2 u_0 + u_0^p \geq 0 \text{ ,in } \Omega \text{ .}$$

(Giga & Kohn 1987 are able to extend the scope of (I1) to include the case $\Omega = \mathbf{R}^N$ and can relax the requirement that $\nabla^2 u_0 + u_0^p \geq 0$ provided p satisfies an upper bound with respect to the space dimension N .)

Similar growth rate estimates are of course available for other nonlinearities f .

Finally, a useful result which may be employed to derive bounds for the solution (or norms of the solution) to problems such as (A) and (B) is an estimate for the gradient of the solution. These types of results are usually difficult to establish (see for example Sperb 1981, Appendix A). For the problem (A), however, Friedman & MacLeod 1985 shows that, provided $U(t_0) = \max_{x\in\Omega} u(x, t_0)$ is sufficiently large, specifically

$$\int_{u_0(x)}^{U(t_0)} f(s)\, ds \geq \tfrac{1}{2}|\nabla u_0(x)|^2 \quad \text{in} \quad \Omega \ ,$$

then for Ω a convex subset of $\mathbf{R}^N$ one has

$$\tfrac{1}{2}|\nabla u(x,t)|^2 \leq \int_{u(x,t)}^{U(t_0)} f(s)\, ds \quad \text{for} \quad x \quad \text{in} \quad \Omega \ , \quad 0 < t < t_0 < T \ .$$

Blow-up results which apply to the more general problem (B) are less widely known. This problem is considered in Chapters 2 and 3 where the type of questions for which answers have been established for the problem (A) are addressed.

In Weissler 1984 and Giga & Kohn 1985 comparisons are made between the solutions to problem (A) and the corresponding ordinary differential equation (D).

In (A) there is a contest between the dissipating affect of the Laplacian and the focussing effect of the nonlinearity $f(u)$. When blow-up occurs it is evident that the nonlinear term dominates. Giga & Kohn, however, observe that the smoothing effect of the Laplacian is still noticeable in the different character of the blow-up. For specific examples Giga & Kohn demonstrate that the blow-up of the partial differential equation (A) is 'flatter' than that of the ordinary differential equation (D).

We anticipate that a similar comparison may be valid between the solutions of (A) and (B).

If the gradient term in (B) is of negative sign then this will have a damping effect and will work against blow-up. In this case, it is not clear if the solutions to problem (B) will exhibit finite-time blow-up when, for the same nonlinearity $f(u)$, the solutions to problem (A) would. In addition a negative gradient term will 'focus' the solution u towards points where the gradient of u is zero and will consequently encourage a more 'peaked' solution. It follows, therefore, that even if the solutions to both (A) and (B) do blow-up, the effect of the gradient term may still be noticed in the form of blow-up or the shape of the solution. Alternatively, if the gradient term is positive, then this will encourage finite-time blow-up. In this case the solutions to problem (B) can be expected to blow-up whenever, for the same nonlinearity $f(u)$, solutions to the problem (A) would. A positive gradient term also has a dissipating effect, however, and may be expected to lead to 'flatter' solutions for the problem (B) than for the corresponding problem (A). This effect may also be noticeable in the form of blow-up which occurs, or in the shape of the solution.

The majority of questions related to the blow-up behaviour of solutions to the problem (B) remain essentially open. Below is a brief summary of those results which are known.

Friedman & Lacey 1988 consider the problem (B) in one-dimension with $G(u,u_x) = uu_x$ and $f(u) = u^p,\ p > 1$, i.e. u is the solution to

$$u_t = u_{xx} + u^p - uu_x \tag{B1}$$

with Dirichlet boundary conditions and positive initial data. For suitable initial data, finite-time blow-up is established for some p , and specifically

(a) finite-time blow-up can occur if $p > 2$, and

(b) there is no blow-up if $p \le 2$.

The result (a) is then extended to nonlinear gradient terms $G(u,u_x)$ and applies to

$$G(u,u_x) = u^k u_x \text{ for large } u \text{ if } p > k+1 ,$$

and

$$G(u,u_x) = u|u_x|^l u_x \text{ for large } u_x \text{ if } 0 \le l \le 2(p-2)/(p+1) .$$

If in addition p is sufficiently large and the initial data is suitable, Friedman & Lacey also show that solutions to the problem (B1) can exhibit single-point blow-up, and bounds on the blow-up rate similar to the inequality (I1) are found. Chipot & Weissler 1987 considers the problem (B) for Ω a bounded subset of

$\mathbf{R}^N$, $f(u) = u^p$, $p > 1$ and $G(u, \nabla u) = |\nabla u|^\beta$, $\beta > 1$, i.e.

$$u_t = \nabla^2 u + u^p - |\nabla u|^\beta \ .$$

In this case, finite-time blow-up is established for suitable initial data provided

$$\beta \leq \frac{2p}{p+1} \ .$$

Bebernes & Eberly 1987 considers a special form of (B) in which $f(u) = e^u$

and $G(u, \nabla u) = |\nabla u|^2$, i.e.

$$u_t = \nabla^2 u + e^u - |\nabla u|^2 \ .$$

Because of the special form of this nonlinearity, quite precise answers to the 'usual' blow-up questions are obtained. This problem may be considered as the limiting case of equality in Chipot & Weissler's condition as $p \to \infty$.

As previously described, because a negative gradient term will work against blow-up, it is natural to expect that, if this term dominates over $f(u)$, then finite-time blow-up may not take place. As in Friedman & Lacey 1988 (or Section 3.2) it can be shown that, if

$$f(u) = G(u, \nabla u) = u^p - u^\alpha |\nabla u|^\beta \ ,$$

then blow-up cannot occur (because there exists a time-independent upper solution to u which is bounded) if $p \leq \alpha + \beta$. This is a generalised form of Friedman & Lacey's result (b).

Further, the techniques used by Friedman & Lacey can be applied (as they are in Section 3.2) to the higher dimensional problem considered by Chipot & Weissler to show that blow-up is possible at least for some $\beta > 2p/(p+1)$.

The aim of Chapter 3 is consequently in the first instance to identify the maximum values of α and β for which blow-up could be possible.

If the term $G(u, \nabla u)$ is less than zero throughout the considered region (so that the presence of this term contributes to the growth of u and encourages blow-up) then very little is known of the blow-up behaviour of solutions. Because in this case blow-up can be established for suitable initial data by comparing the solution of problem (B) with the corresponding solution of problem (A) (see Section 3.5), interest in this area is concentrated on identifying the form of blow-up and the blow-up set.

An indication of what behaviour may occur in this case has been highlighted by Kawohl. Kawohl has observed that, if

$$u_t = \nabla^2 u + u^p + |\nabla u|^2 \ ,$$

i.e. if $f(u) = u^p$ and $G(u,\nabla u) = -|\nabla u|^2$ in (B), with Dirichlet boundary conditions and positive initial data, then

$$v_t = \nabla^2 v + (1 + v)[\log(1 + v)]^p$$

where $v = e^u - 1$. Known results for the blow-up behaviour of the problem (A) then apply to v and show that $p > 1$ is required for the existence of blow-up. In addition, the combined results of Friedman & MacLeod 1985 and Lacey 1986 show that in the radically symmetric case single-point blow-up occurs if $p > 2$, regional blow-up if $p = 2$, and global blow-up if $1 < p < 2$.

This example suggests that there is a direct linear relationship between the opposing effects of the nonlinear term $f(u)$ and the power of the gradient term in $G(u,\nabla u)$ in determining the shape of the solution and the form of blow-up for problems like (B).

P.3 Comparison techniques and the maximum principle

Many of the arguments applied in this work rely on application of the maximum principles for elliptic and parabolic equations. These in turn form the basis of many useful comparison techniques (Protter & Weinberger 1967, Sperb 1981, Sattinger 1972, Amann 1971 & 1976b).

In the parabolic case, the maximum principle asserts that, if u is a nonconstant solution to

$$u_t = \mathcal{L}u + h(x,t)u \quad \text{in} \quad \Omega x(0,T) \tag{P1}$$

where Ω is a bounded domain in $\mathbf{R}^N$ with smooth boundary $\partial\Omega$, T is finite, $\mathcal{L}$ is a uniformly elliptic operator with bounded coefficients, $h(x,t) \le 0$ within $\Omega x(0,T)$, and u has a positive interior maximum, M , within $\Omega x(0,T)$, then $u = M$ at some point (x_m, t_m) with $t_m = 0$ or $x_m \in \partial\Omega$

Further, the requirement that $h(x,t) \le 0$ can be relaxed if $M = 0$ and $h(x,t)$ is bounded from above .

(Protter & Weinberger 1967, Sperb 1981).

The strong maximum principle for elliptic equations of E. Hopf has been extended to parabolic equation by Friedman 1964. This can be stated as :-

Let u be a nonconstant solution to (P1) and suppose the positive maximum of u is attained at some point, x_m say, of $\partial\Omega x(0,T)$. Then if $\partial\Omega$ has the 'interior sphere property' at x_m we have that $\frac{\partial u}{\partial \nu} > 0$ at x_m for any vector ν which points outward at x_m .

The boundary $\partial\Omega$ is said to have the interior sphere property at x_m if there exists a sphere, S , of radius $r_0 > 0$ contained within Ω such that $S \cap \partial\Omega = x_m$ (see for example Sperb 1981, Protter & Weinberger 1967).

The corresponding minimum principles are obtained by replacing u by $-u$.

These are the two 'standard' maximum principles for parabolic equations and are used extensively in this work. Maximum principles are also applied to suitable functionals of the solutions to problems (A) and (B), and this follows by showing that the functional conforms to the requirements of the appropriate 'standard' maximum principle.

The maximum principles for elliptic equations are similarly defined (but require that $h(x, t) \leq 0$ in (P1)) and can be found in Protter & Weinberger 1967 and Sperb 1981.

One slight modification of the standard maximum principles as quoted above which can be useful is that the requirement that the coefficients of $\mathcal{L}$ are bounded and that h is bounded from above can be relaxed at the expense of allowing a maximum to be sited at one of the unbounded points. Essentially, therefore, this means that the conclusion of the maximum principle can be augmented to read 'a positive maximum of u must be attained at $t = 0$, or on $\partial\Omega$, or at an interior point in Ω where the coefficients of $\mathcal{L}$ become unbounded or, if $M = 0$, where h is unbounded from above'. This variation of the maximum principle follows from first principles and is also used in Sperb 1981. Much of the work of Chapters 1-3 is concerned with finding upper and lower solutions to the solutions and to functions of the solutions of both problems (A) and (B). Generally, upper and lower solutions are defined for any solution to a problem for which the maximum principle can be applied. As an example, therefore, an upper solution for the problem (A) is a function, say $\overline{u}(x,t)$ satisfying

$$\frac{\partial \overline{u}}{\partial t} \geq \nabla^2 \overline{u} + \lambda f(\overline{u}) \qquad x \text{ in } \Omega,\ t > 0$$

$$\overline{u}(x,0) \geq u_0(x) \qquad x \text{ in } \Omega$$

$$\frac{\partial \overline{u}}{\partial n} + \beta \overline{u} \geq 0 \qquad x \text{ on } \partial\Omega,\ t > 0\ .$$

A lower solution, say $\underline{u}(x,t)$, is defined by reversing the inequalities above.

The maximum principle may be used to establish that if there exists a solution u and upper and lower solutions to the problem (A) then we must have

$$\underline{u} \leq u \leq \overline{u}\ .$$

For the problem (A), one also has a complementary existence result in that, if there exists upper and lower solutions with $\underline{u} \le \overline{u}$ for $0 \le t \le T$ with $T > 0$,

then there exists a solution u and

$$\underline{u} \le u \le \overline{u} \text{ for } (x,t) \in \Omega x(0,T) \quad \text{(see Sattinger 1972).}$$

Upper and lower solutions are analogously defined in the case of elliptic problems.

Chapter 1 Estimate of Blow-up Time

1.1. Introduction.

Throughout this chapter we consider the parabolic initial-value problem

$$u_t = \nabla^2 u + \lambda e^u \qquad \text{in } \Omega,\ t>0, \tag{1.1.1}$$

$$u(x,0) = u_0(x) \qquad \text{in } \Omega, \tag{1.1.2}$$

$$\frac{\partial u}{\partial n} + \beta u = 0 \qquad \text{on } \partial\Omega,\ t>0, \tag{1.1.3}$$

where Ω is a bounded region in $\mathbf{R}^N$ with smooth boundary $\partial\Omega$, and

$$0 < \beta(x) \le \infty.$$

The analysis is restricted to the particular case of (1.1.1) although it is felt that similar results should be available for any nonlinearity $f(u)$ for which

$$f(u), f'(u) \text{ and } f''(u) > 0 \text{ for all } u .$$

One characteristic of the solutions to problems such as (1.1.1)-(1.1.3) which is of particular interest is that of thermal runaway or finite-time blow up. Finite time blow up occurs if the solution becomes infinite at some point(s) of Ω in a finite time.

Specifically, if we assume that

$$u_0 \in C^1(\Omega), \text{ with } u_0 \ge 0 \text{ in } \Omega \text{ and } \frac{\partial u_0}{\partial n} + \beta u_0 = 0 \text{ on } \partial\Omega \tag{1.1.4}$$

then (1.1.1)-(1.1.3) has a unique solution for $t<\sigma_0$ if σ_0 is sufficiently small.

Further, the maximum principle may be applied to establish that

$$u(x,t) \geq 0$$

and, if $\nabla^2 u_0 + \lambda e^{u_0} \geq 0$ for x in Ω , then

$$U(t) = \max_{x \in \bar{\Omega}} u(x,t)$$

is monotone increasing in t .

If a solution exists for all $t < \sigma$ with $\sigma < \infty$, and $U(\sigma - 0) < \infty$, then the solution can be uniquely continued into some interval $0 < t < \sigma + \epsilon$ for some $\epsilon > 0$.

Hence, if T denotes the supremum of all σ such that the solution exists for all $t < \sigma$, then

$$U(T-0) = \infty$$

and $T < \infty$ characterises finite-time blow up of u .

Although a widely studied problem, the particular aim of this section is derived directly from the analysis of Lacey 1983 in which (1.1.1)-(1.1.3) and the case of more general $f(u)$ are discussed in detail.

For the problem (1.1.1)-(1.1.3) and the more general nonlinearities considered in Lacey 1983 it is known that there exists some value of λ, λ^* say, such that for $\lambda<\lambda^*$ there is a solution to the steady state problem whereas if $\lambda>\lambda^*$ no steady-state solution exists. In the case of (1.1.1)-(1.1.3) the steady- state solution, $\omega(x)$, satisfies

$$\nabla^2\omega + \lambda e^{\omega} = 0 \qquad \text{in } \Omega \tag{1.1.7}$$

with

$$\frac{\partial\omega}{\partial n} + \beta\omega = 0 \qquad \text{on } \partial\Omega\ . \tag{1.1.8}$$

Lacey 1983 successfully established finite time blow up of u for $\lambda>\lambda^*$ irrespective of u_0 for a range of functions f (including the exponential case) and regions Ω .

In addition, and of particular interest here, the following results were also established.

It was shown that when $\lambda=\lambda^*$ leads to a solution, ω^* , to the steady-state problem (1.1.7)-(1.1.8) there exists a finite time

$$t = t_u(\lambda-\lambda^*)^{-1/2} \tag{1.1.9}$$

by which the solution to (1.1.1)-(1.1.3) must have blown up.

Furthermore, if $u_0 < \omega^*$, then there also exists a time

$$t = t_1(\lambda-\lambda^*)^{-1/2} \tag{1.1.10}$$

at which u still exists.

Hence, the blow up time of u, t_b say, must lie somewhere between these estimates.

Motivated by these bounds on t_b , a formal asymptotic estimate is derived for $\lambda-\lambda^*$ small and provided

$$\text{either} \quad \frac{\partial u_0}{\partial n} > \frac{\partial \omega^*}{\partial n} \quad \text{or} \quad u_0 < \omega^* \quad \text{on} \quad \partial\Omega \ . \tag{1.1.11}$$

This analysis indicates that

$$t_b \sim \tau_b(\lambda-\lambda^*)^{-1/2} \quad \text{as} \quad \lambda\to\lambda^*+0 \tag{1.1.12}$$

with an explicit expression obtained for τ_b .

The aim of this chapter is to strengthen this particular result by deriving, through the use of upper and lower solutions, bounds on the blow up time of u which are of the form described by (1.1.9) and (1.1.10) but which are closer to the asymptotic estimate of (1.1.12).

In essence, therefore, this analysis represents an extension of the techniques and arguments applied in Lacey 1983 to develop the estimates (1.1.9), (1.1.10) and (1.1.12). Theses techniques are consequently first briefly outlined in Section 1.2. Section 1.3 then establishes the desired upper bound for the blow up time of u by constructing a lower solution to u which exhibits finite time blow up. By comparison u must also blow up prior to this time.

Section 1.4 establishes a complimentary lower bound for the blow up time of u which remains finite up to some time. Hence by comparison, the blow up of u must occur at some later time.

On combining the estimates derived in Sections 1.3 and 1.4 we are then able to bound the blow up time of u 'close to' the asymptotic estimate of (1.1.12).

Section 1.2 Summary of previous results

1.2.1 Derivation of the upper bound of (1.1.9)

If λ^* is in the spectrum of the steady-state problem (1.1.7)-(1.1.8) we denote by ω^* this steady-state solution. Hence,

$$\nabla^2\omega^* + \lambda^* e^{\omega^*} = 0 \quad \text{in} \quad \Omega \tag{1.2.1}$$

with

$$\frac{\partial\omega^*}{\partial n} + \beta\omega^* = 0 \quad \text{on} \quad \partial\Omega \tag{1.2.2}$$

and there exists an eigen solution φ^* such that

$$\nabla^2\varphi^* + \lambda^* e^{\omega^*}\varphi^* = 0 \quad \text{in} \quad \Omega \tag{1.2.3}$$

$$\frac{\partial\varphi^*}{\partial n} + \beta\varphi^* = 0 \quad \text{on} \quad \partial\Omega\ , \tag{1.2.4}$$

with

$$\varphi^* > 0 \quad \text{in} \quad \Omega \quad \text{and} \quad \int_\Omega \varphi^* = 1\ . \tag{1.2.5}$$

Denoting by $v(x,t)$ the function

$$v(x,t) = u(x,t) - \omega^*(x) \quad \text{in} \quad \Omega,\ t>0\ , \tag{1.2.6}$$

we find that v satisfies

$$v_t = (\lambda-\lambda^*)e^u + \lambda^*[e^v-(1+v)]e^{\omega^*} + \lambda^* v e^{\omega^*} + \nabla^2 v \quad \text{in} \quad \Omega,\ t>0, \tag{1.2.7}$$

with

$$\frac{\partial v}{\partial n}+\beta v=0 \quad \text{on} \quad \partial\Omega . \tag{1.2.8}$$

If $a^*(t)$ is defined as

$$a^*(t) = \int_\Omega \varphi^* v \, dx \tag{1.2.9}$$

then by multiplying both sides of equation (1.2.7) by φ^* and integrating over Ω we see that

$$a^*{}_t = (\lambda-\lambda^*)\int_\Omega \varphi^* e^{u} dx + \lambda^*\int_\Omega \varphi^* \{e^{v}-(1+v)\} e^{\omega^*} dx. \tag{1.2.10}$$

From the maximum principle applied to (1.1.1)-(1.1.3) we have that

$$u(x,t) \ge \min\left(0, \inf_{x\in\Omega} u_0(x)\right) = \overline{u}_0 \quad \text{say} \tag{1.2.11}$$

and hence that

$$\int_\Omega \varphi^* e^{u} dx \ge e^{\overline{u}_0}\int_\Omega \varphi^* dx = e^{\overline{u}_0}. \tag{1.2.12}$$

The inequality (1.2.11) holds because, if u attains a negative minimum at some $t\neq 0$, then this minimum must be attained at some $x\in\partial\Omega$. However, from the boundary condition (1.1.3), $\frac{\partial u}{\partial n} + \beta u = 0,$ so that $\frac{\partial u}{\partial n} = -\beta u > 0$ at this point.

It follows that there exists a smaller value of u within Ω which establishes a contradiction.

To continue, the term $(e^{v} - (1+v))e^{\omega^*}$ in the second integrand of (1.2.10) satisfies

$$(e^{v} - (1+v))e^{\omega^*} \geq \tfrac{1}{2}v^2 e^{\omega^*} \geq \tfrac{1}{2}v^2 \qquad \text{if} \quad v \geq 0 , \tag{1.2.13}$$

and
$$(e^{v} - (1+v))e^{\omega^*} \geq \tfrac{1}{2}v^2 e^{(\omega^* + v)} = \tfrac{1}{2}v^2 e^{u} \qquad \text{if} \quad v \leq 0 . \tag{1.2.14}$$

It follows from (1.2.13) and (1.2.14), that

$$(e^{v} - (1+v))e^{\omega^*} \geq \tfrac{1}{2}v^2 . \min[1, e^{u}] . \tag{1.2.15}$$

Finally, we see from (1.2.11) that

$$e^{u} \geq \exp \overline{u}_0$$

and hence that

$$(e^{v} - (1+v))e^{\omega^*} \geq \tfrac{1}{2}v^2 k \tag{1.2.16}$$

where
$$k = \begin{cases} 1 & \text{if} \quad \inf_{x\in\Omega} u_0(x) \geq 0 \\ \exp\left\{\inf_{x\in\Omega} u_0(x)\right\} & \text{if} \quad \inf_{x\in\Omega} u_0(x) < 0 \end{cases}$$

Substituting the estimates (1.2.12) and (1.2.16) into equation (1.2.10) yields that

$$a_t^* \geq (\lambda - \lambda^*)e^{\overline{u}_0} + \lambda^* \int_\Omega \tfrac{1}{2}k\varphi^* v^2 dx = (\lambda - \lambda^*)e^{\overline{u}_0} + \tfrac{1}{2}k\lambda^* \int_\Omega \varphi^* v^2 dx .$$

But, $\int_\Omega \varphi^* v^2 dx \geq \left[\int_\Omega \varphi^* v \, dx \right]^2 = a^{*2}$

by Jensen's inequality, so that

$$a_t^* \geq k_1 + k_2 a^{*2} \qquad \text{for} \quad t>0, \tag{1.2.17}$$

where

$$k_1 = (\lambda-\lambda^*)e^{\bar{u}_0} = (\lambda-\lambda^*).\min\left[1,\ \exp\left\{\inf_{x\in\Omega} u_0(x)\right\}\right] \tag{1.2.18}$$

and $$k_2 = \tfrac{1}{2}k\lambda^* = \tfrac{1}{2}\lambda^* = \begin{cases} 1 & \text{if } \inf_{x\in\Omega} u_0(x) \geq 0 \\ \exp\left\{\inf_{x\in\Omega} u_0(x)\right\} & \text{if } \inf_{x\in\Omega} u_0(x) < 0 \end{cases} \tag{1.2.19}$$

Hence,

$$a^*(t) \geq \left[\frac{k_1}{k_2}\right]^{\frac{1}{2}} \tan\left\{(k_1k_2)^{\frac{1}{2}}t - \frac{\pi}{2}\right\}.$$

We conclude, therefore, that u must cease to exist at some time t_b where

$$t_b \leq t_u(\lambda - \lambda^*)^{-\frac{1}{2}} \tag{1.2.20}$$

and $$t_u = \frac{\pi(\lambda - \lambda^*)^{\frac{1}{2}}}{(k_1k_2)^{\frac{1}{2}}} = \frac{\pi}{(\tfrac{1}{2}\lambda^* e^{\bar{u}_0}k)}$$

with $\bar{u}_0 = \exp\left\{\inf_{x\in\Omega} u_0(x)\right\}$ and k as in (1.2.19).

1.2.2 Derivation of the lower bound of (1.1.10)

In order to derive the lower bound of (1.1.10) we assume that

$$u_0(x) < \omega^*(x) \quad \text{in} \quad \Omega, \; t>0, \tag{1.2.21}$$

with

$$\frac{\partial u_0}{\partial n} > \frac{\partial \omega^*}{\partial n} \quad \text{or} \quad u_0 < \omega^* \quad \text{on} \quad \partial\Omega \; . \tag{1.2.22}$$

We define the function $u^*(x,t)$ to be the solution to

$$u_t^* = \nabla^2 u^* + \lambda^* e^{u^*} \qquad \text{in} \quad \Omega, \; t>0, \tag{1.2.23}$$

$$u^*(x,0) = u_0(x) \qquad \text{in} \quad \Omega \; , \tag{1.2.24}$$

with $$\frac{\partial u^*}{\partial n} + \beta u^* = 0 \qquad \text{on} \quad \partial\Omega, \; t>0 \; . \tag{1.2.25}$$

We also define $\hat{u}(x,t)$ as the difference between u^* and ω^* , i.e.

$$u^* = \omega^* - \hat{u}. \tag{1.2.26}$$

Hence, $$\hat{u}_t = -\lambda^* \exp(\omega^* - \hat{u}) + \nabla^2 \hat{u} - \nabla^2 \omega^*, \qquad \text{in} \quad \Omega, \; t>0, \tag{1.2.27}$$

with $$\hat{u}(x,0) = \omega^*(x) - u_0(x), \qquad \text{in} \quad \Omega, \tag{1.2.28}$$

and $$\frac{\partial \hat{u}}{\partial n} + \beta \hat{u} = 0 \qquad \text{on} \quad \partial\Omega, \; t>0. \tag{1.2.29}$$

The maximum principle applied to (1.2.27)-(1.2.29) establishes that a negative minimum of $\hat{u}$ must occur either on $\partial\Omega$, or at $t = 0$.

From the condition (1.2.29), however, a negative minimum of $\hat{u}$ at some point $(\bar{x},\bar{t})$ with $\bar{x}\in\partial\Omega$ would require that

$$\frac{\partial \hat{u}}{\partial n} > 0 \qquad \text{at} \quad (\bar{x},\bar{t}),$$

and hence that there exists some $x\in\Omega$ with

$$\hat{u}(x,\bar{t}) < \hat{u}(\bar{x},\bar{t})$$

which would lead to a contradiction.

Further, either of the alternatives of (1.2.22) also ensure that if a negative minimum of $\hat{u}$ occurs at $t = 0$, then this minimum must be located at some point within Ω .

However, the condition (1.2.21) prohibits a negative minimum of $\hat{u}$ from occurring within Ω at $t=0$ and hence establishes that

$$\hat{u}(x,t) \geq 0 \qquad \text{throughout} \quad \Omega, \; t > 0. \tag{1.2.30}$$

From (1.2.27) we see that

$$\hat{u}_t = \lambda^* e^{\omega^*} [\, 1 - e^{-\hat{u}} \,] + \nabla^2 \hat{u}$$

and following (1.2.13), (1.2.14) we get the estimate

$$1 - e^{-\hat{u}} \geq \hat{u} - \tfrac{1}{2}\hat{u}^2,$$

so that

$$\hat{u}_t \geq \lambda^* e^{\omega^*} [\, \hat{u} - \tfrac{1}{2}\hat{u}^2 \,] + \nabla^2 \hat{u} \qquad \text{in} \quad \Omega, \ t > 0. \tag{1.2.31}$$

Hence, if we now define the constant k_1 as

$$k_1 = \tfrac{1}{2}\lambda^* \sup_{x \in \Omega} e^{\omega^*}$$

then $$\hat{u}_t \geq -\, k_1 \hat{u}^2 + \lambda^* \hat{u} e^{\omega^*} + \nabla^2 \hat{u}, \qquad \text{in} \quad \Omega, \ t > 0. \tag{1.2.32}$$

We now look for a function $\psi(x,t)$ which will be a lower solution to $\hat{u}$ and choose

$$\psi(x,t) = \frac{k_2 \varphi^*(x)}{(t + t_0)} \tag{1.2.33}$$

where k_2 and t_0 are positive constants to be chosen as appropriate and $\varphi^*(x)$ is taken as the solution to (1.2.3), (1.2.4) satisfying

$$\varphi^* > 0 \quad \text{in} \quad \Omega \quad \text{and} \quad \int_\Omega \varphi^{*2} = 1. \tag{1.2.34}$$

With this ψ, we see that

$$\psi_t + k_1 \psi^2 - \lambda^* \psi e^{\omega^*} - \nabla^2 \psi = \frac{-\, k_2 \varphi^*}{(t + t_0)^2} [1 - k_1 k_2 \varphi^*]$$

and the right hand side of this equation will be less than or equal to zero if we set

$$k_2 = \frac{1}{k_1 \sup_{x\in\Omega} \varphi^*(x)}. \tag{1.2.35}$$

Further,

$$\psi(x,0) - \hat{u}(x,0) = \frac{k_2\varphi^*}{t_0} - (\omega^* - u_0) \le 0$$

provided t_0 is chosen such that

$$t_0 \ge k_2 \cdot \sup_{x\in\Omega}\left[\frac{\varphi^*}{\omega^*-u_0}\right]. \tag{1.2.36}$$

It follows, therefore, that k_2 and t_0 chosen to satisfy conditions (1.2.35) and (1.2.36) would establish $\psi(x,t)$ as defined by (1.2.33) as a lower solution to $\hat{u}$.

We next consider the function $u_1(x,t)$ defined as

$$u_1(x,t) = u(x,t) - u^*(x,t) \quad \text{in } \Omega,\ t > 0. \tag{1.2.37}$$

Hence,

$$\begin{aligned} u_{1t} &= (\lambda - \lambda^*) e^{u} + \lambda^*[e^{u} - e^{u^*}] + \nabla^2 u_1 \\ &= (\lambda - \lambda^*) e^{u} + \lambda^* e^{u^*}[e^{u_1} - 1] + \nabla^2 u_1 \end{aligned} \quad \text{in } \Omega,\ t > 0 \tag{1.2.38}$$

with $\quad u_1(x,0) = 0 \qquad$ in $\Omega,$ (1.2.39)

and $\quad \dfrac{\partial u_1}{\partial n} + \beta u_1 = 0 \qquad$ on $\partial\Omega,\ t > 0.$ (1.2.40)

As $\lambda > \lambda^*$, the maximum principle applied to (1.2.38)-(1.2.39) yields that

$$u_1(x,t) \geq 0 \qquad \text{throughout } \Omega,\ t > 0. \tag{1.2.41}$$

We shall assume, however, that

$$u = u^* + u_1 < \omega^*. \tag{1.2.42}$$

Hence, $u^* < \omega^* - u_1$

which, when substituted in (1.2.38) yields that

$$u_{1t} \leq (\lambda - \lambda^*) e^{u} + \lambda^* e^{(\omega^* - u_1)} \left[e^{u_1} - 1 \right] + \nabla^2 u_1 \tag{1.2.43}$$

because $e^{u_1} - 1 \geq 0$ as a result of (1.2.41).

As a further consequence of (1.2.41),

$$e^{-u_1} \left[e^{u_1} - 1 \right] = \left[1 - e^{-u_1} \right] \leq u_1$$

and (1.2.43) becomes

$$u_{1t} \leq (\lambda - \lambda^*) e^{u} + \lambda^* e^{\omega^*} u_1 + \nabla^2 u_1 .$$

Again, in light of (1.2.42), we see that

$$u_{1t} \leq (\lambda - \lambda^*) e^{\omega^*} + \lambda^* e^{\omega^*} u_1 + \nabla^2 u_1 \qquad \text{in } \Omega,\ t > 0, \tag{1.2.44}$$

provided condition (1.2.42) holds.

We now look to define a function, $\psi_1(x,t)$ say, which will be an upper solution to u_1 .

We first choose the function $\Phi^*(x)$ to satisfy

$$\nabla^2\Phi^* + \lambda^* e^{\omega^*}\Phi^* + e^{\omega^*} = \varphi^* I_1 = 0 \quad \text{in} \quad \Omega \tag{1.2.45}$$

with $$\frac{\partial\Phi^*}{\partial n} + \beta\Phi^* = 0 \quad \text{on} \quad \partial\Omega \tag{1.2.46}$$

where $$I_1 = \int_\Omega \varphi^* e^{\omega^*} dx \tag{1.2.47}$$

and $\Phi^* \geq 0$ throughout Ω.

We next choose $\psi_1(x,t)$ such that

$$\psi_1(x,t) = (\lambda - \lambda^*)(I_1\varphi^* t + \Phi^*) . \tag{1.2.48}$$

Now

$$\psi_{1t} - (\lambda - \lambda^*)e^{\omega^*} - \lambda^* e^{\omega^*}\psi_1 - \nabla^2\psi_1 = 0 \quad \text{in} \quad \Omega,\ t > 0$$

and $$\psi_1(x,0) = (\lambda - \lambda^*)\Phi^*(x) \geq 0 \quad \text{in} \quad \Omega$$

with $$\frac{\partial\psi_1}{\partial n} + \beta\psi_1 = 0 \quad \text{on} \quad \partial\Omega,\ t > 0 .$$

It follows, therefore, that $\psi_1(x,t)$ as defined by (1.2.48) is an upper solution to u_1 provided inequality (1.2.44) is valid, i.e. provided $u < \omega^*$.

On combining this solution to $\hat{u}$ and upper solution to u_1 we see that

$$u = w^* - \hat{u} + u_1$$

and hence that

$$u(x,t) \le \omega^*(x) - \psi(x,t) + \psi_1(x,t) \le \omega^*(x) \tag{1.2.49}$$

provided $\psi(x,t) - \psi_1(x,t) \ge 0.$ (1.2.50)

The condition (1.2.50) will be satisfied if

$$\psi = \frac{k_2\varphi^*}{(t + t_0)} \ge (\lambda - \lambda^*)\ [\ I_1\varphi^* t + \Phi^*\ ] = \psi_1$$

i.e. if

$$k_2 \ge (\lambda - \lambda^*)(t + t_0)\ [\ I_1 t + A\ ] \tag{1.2.51}$$

where $A = \sup_{x\in\Omega}\left(\frac{\Phi^*}{\varphi^*}\right)$.

The condition (1.2.51) requires that

$$t^2(\lambda - \lambda^*)I_1 + t(\lambda - \lambda^*)[I_1 t_0 + A] - k_2 + A(\lambda - \lambda^*)t_0 \le 0 \tag{1.2.52}$$

If λ is sufficiently close to λ^*, inequality (1.2.52) requires, to highest order, that

$$-\left[\frac{k_2}{I_1}\right]^{\frac{1}{2}}(\lambda - \lambda^*)^{-\frac{1}{2}} - O(1) \le t \le \left[\frac{k_2}{I_1}\right]^{\frac{1}{2}}(\lambda - \lambda^*)^{-\frac{1}{2}} - O(1)$$

For $t \ge 0$ inequality (1.2.52) will always be satisfied if

$$t \le t_1(\lambda - \lambda^*)^{-\frac{1}{2}} \tag{1.2.53}$$

where $t_1 = \frac{1}{2}\left[\frac{k_2}{I_1}\right]^{\frac{1}{2}}$ (1.2.54)

We conclude, therefore, that the estimate (1.2.49) holds for all t satisfying (1.2.53) and hence that t_b, the blow up time of u, must satisfy

$$t_b > t_1(\lambda - \lambda^*)^{-\frac{1}{2}}. \tag{1.2.55}$$

1.2.3 An asymptotic estimate of t_b

Motivated by the bounds on t_b described in Sections 1.2.1 and 1.2.2, Lacey 1983 proceeds to establish an estimate of t_b as an asymptotic expression for $\epsilon = (\lambda - \lambda^*)$ in the limit $\lambda \to \lambda^*$.

This analysis is very similar to that which will be applied in Sections 1.3 and 1.4 in an attempt to derive 'better' upper and lower bounds on t_b. To avoid repetition, therefore, the asymptotic analysis of Lacey 1983 is only briefly outlined below.

The upper solution to u derived in Section 1.2.2 and the examination of the function $v(x,t) = u(x,t) - \omega^*(x)$ of Section 1.2.1 suggest that three regimes of time should be considered. In time regime I $u < \omega^*$ and the upper solution to u established in 1.2.2 suggests that u be considered as the expansion

$$u \sim u^* + \epsilon u_2 + \ldots . \qquad \text{for} \quad \epsilon \to 0. \tag{1.2.56}$$

Further, this characterisation is expected to remain valid while t is of order 1.

Hence, the leading term in this expansion, u^*, satisfies

$$u_t^* = \nabla^2 u^* + \lambda^* e^{u^*} \quad \text{in} \quad \Omega, t > 0,$$

with $\quad u^*(x,0) = u_0(x) \quad$ in Ω,

and $\quad \dfrac{\partial u^*}{\partial n} + \beta u^* = 0 \qquad$ on $\partial\Omega, t > 0$.

As in Section 1.2.2, we require that $u_0(x)$ satisfy the conditions (1.2.21), and (1.2.22) in light of which we see that $u^*(x,t) \le \omega^*(x)$, and that $u^* \to \omega^*$ as $t \to \infty$.

If we write $u^* = \omega^* - v$, , then for large t , v will become small and to highest order,

$$v_t = \nabla^2 v + \lambda^* e^{\omega^*}\{v - \tfrac{1}{2}v^2 + \ldots .\} \quad \text{in} \quad \Omega .$$

It follows, that if $v(x,t)$ is chosen as

$$v(x,t) = \varphi^*(x)\, v(t) + v_2(x,t) + \ldots .$$

then $\quad \varphi^* v_t = -\tfrac{1}{2}\lambda^* e^{\omega^*}\varphi^{*2}v^2 + \nabla^2 v_2 + \lambda^* e^{\omega^*} v_2 + \ldots .$

Hence,

$$v_t\int_\Omega \varphi^{*2}dx = -\tfrac{1}{2}v^2\lambda^*\int_\Omega \varphi^{*3}e^{\omega^*}dx$$

to allow the existence of some v_2 ,

and

$$v(t) \sim \frac{1}{\tfrac{1}{2}\lambda^* I_3 t} \qquad \text{for large} \quad t \tag{1.2.57}$$

where $$I_3 = \int_\Omega \varphi^{*^3} e^{\omega^*} dx, \tag{1.2.58}$$

and φ^* is the solution to (1.2.3), (1.2.4) with (1.2.34), i.e.

$$\varphi^* > 0 \quad \text{in} \quad \Omega \quad \text{and} \quad \int_\Omega \varphi^{*^2} = 1.$$

From (1.2.57) it follows that,

$$u^*(x,t) \sim \omega^*(x) - \frac{\varphi^*(x)}{\tfrac{1}{2}\lambda^* I_3 t} \quad \text{for large} \quad t. \tag{1.2.59}$$

In time regime II, t is of order $\epsilon^{-\frac{1}{2}}$, and Sections 1.2.1 and 1.2.2 suggest that

$$u \sim \omega^* + \epsilon^{\frac{1}{2}} u_1 + \epsilon u_2 + \ldots. \quad \text{as} \quad \epsilon \to 0 .$$

We rescale t in terms of the order one variable τ such that $t = \epsilon^{-\frac{1}{2}}\tau$ in which case (1.1.1) yields

$$\begin{aligned} \epsilon^{\frac{1}{2}} \{ \epsilon^{\frac{1}{2}} u_{1\tau} + \epsilon u_{2\tau} + \ldots \} = (\lambda^* + \epsilon)(1 + \epsilon^{\frac{1}{2}} u_1 + \epsilon u_2 + \ldots.) e^{\omega^*} \\ + \nabla^2 \omega^* + \epsilon^{\frac{1}{2}} \nabla^2 u_1 + \epsilon \nabla^2 u_2 + \ldots. \end{aligned}$$

$$\text{as} \quad \epsilon \to 0. \tag{1.2.60}$$

By considering the equations relating the highest order terms in (1.2.60) in turn, it can be established that

$$u_1(x,\tau) = \varphi^*(x) a_1(\tau)$$

where

$$a_{1\tau}(\tau) = \tfrac{1}{2}\lambda^* I_3 a_1^2(\tau) + I_1 \tag{1.2.61}$$

and

$$I_1 = \int_\Omega \varphi^* e^{\omega^*} dx. \tag{1.2.62}$$

Hence,

$$a_1(\tau) = \left[\frac{I_1}{\tfrac{1}{2}\lambda^* I_3}\right]^{\frac{1}{2}} \tan\left\{(\tfrac{1}{2}\lambda^* I_1 I_3)^{\frac{1}{2}}\tau - C\right\}$$

and the choice $C = \frac{\pi}{2}$ so that

$$a_1(\tau) = \left[\frac{I_1}{\tfrac{1}{2}\lambda^* I_3}\right]^{\frac{1}{2}} \tan\left\{(\tfrac{1}{2}\lambda^* I_1 I_3)^{\frac{1}{2}}\tau - \frac{\pi}{2}\right\} \tag{1.2.63}$$

will ensure this solution matches with that of time regime I (i.e. (1.2.59)).

The characterisation of time regime II fails as $t = \epsilon^{-\frac{1}{2}}\tau$ approaches $t^+ = \epsilon^{-\frac{1}{2}}\tau^+$ say, where

$$\tau^+ = \frac{\pi}{[\tfrac{1}{2}\lambda^* I_1 I_3]^{\frac{1}{2}}}. \tag{1.2.64}$$

In this region, (1.2.63) suggests that we again look for $u \sim u^* + \epsilon u_2 + \ldots$ as $\epsilon \to 0$ although in this region, which we call time regime III, u now satisfies

$u > \omega^*$. Letting $t = t^* + \hat{t}$ for $\hat{t} < 0$, we again see that the leading term,

u^*, satisfies

$$\frac{\partial u^*}{\partial t} = \nabla^2 u^* + \lambda^* e^{u^*} \qquad \text{in } \Omega, t > 0$$

with $$\frac{\partial u^*}{\partial n} + \beta u^* = 0 \qquad \text{on } \partial\Omega, t > 0.$$

In this case, however, we expect that $u^* \to \infty$ as $\hat{t} \to \hat{t}_b$ and as $t \to t^* + \hat{t}_b$.

If we now write $u^* = \omega^* + v$,

we see that if

$$v(x,t) = \varphi^* v(t) + v_2(x,t) + \ldots . \qquad \text{for } -\hat{t} >> 1$$

then $$v_t \int_\Omega \varphi^{*2} = \tfrac{1}{2}\lambda^* v^2 \int_\Omega \varphi^* e^{\omega^*} dx \tag{1.2.65}$$

to allow the existence of some v_2 .

Hence, as $v(t^*) \sim +\infty$, (1.2.65), when integrated from t to t^*, establishes

that

$$v(t) \sim -\frac{1}{\tfrac{1}{2}\lambda^* I_3 \hat{t}} \quad \text{for } -\hat{t} >> 1$$

so that

$$u^* \sim \omega^* - \frac{\varphi^*}{\frac{1}{2}\lambda^* I_3 \hat{t}} \qquad \text{as} \quad \hat{t} \to -\infty. \tag{1.2.66}$$

The preceding analysis establishes t^+ as an estimate for t_b in the limit $\lambda \to \lambda^*$ so that

$$t_b \sim t^+ = e^{-\frac{1}{2}\tau^+} = \frac{\pi\,(\lambda-\lambda^*)^{-\frac{1}{2}}}{\left[\frac{1}{2}\lambda^* I_1 I_3\right]^{\frac{1}{2}}} \qquad \text{as} \quad \lambda \to \lambda^*. \tag{1.2.67}$$

Section 1.3 An upper bound on t_b

1.3.1 A lower solution in time region I

The purpose of this section is to derive a lower solution to u while t is within time regime I. The asymptotic analysis of Lacey 1983 (described in Section 1.2.3) indicates that

$$u \sim \omega^* - \frac{\varphi^*}{\frac{1}{2}\lambda^* I_3 t} \quad \text{as} \quad t \to \infty$$

in this region and we look for a lower solution to u which exhibits this behaviour.

As in Section 1.2.2 we begin by considering the function $u^*(x, t)$ satisfying (1.2.23)-(1.2.25) so that

$$u_t^* = \nabla^2 u^* + \lambda^* e^{u^*} \quad \text{in} \quad \Omega,\ t > 0,$$

$$u^*(x,0) = u_0(x) \quad \text{in} \quad \Omega,$$

$$\frac{\partial u^*}{\partial n} + \beta u^* = 0 \quad \text{on} \quad \partial\Omega,\ t > 0,$$

and we again require that the initial condition $u_0(x)$ satisfy the conditions (1.2.21) and (1.2.22).

It follows, as $\lambda > \lambda^*$, that by comparison

$$u(x,t) \geq u^*(x,t) \qquad \text{for} \quad x \in \Omega, \; t > 0. \tag{1.3.1}$$

If we again describe by $\hat{u}(x,t)$ the difference between $u^*(x,t)$ and $\omega^*(x)$, so that

$$\hat{u}(x,t) = \omega^*(x) - u^*(x,t)$$

then $\hat{u}(x,t)$ satisfies (1.2.27)-(1.2.29), i.e.

$$\hat{u}_t = \nabla^2 \hat{u} - \nabla^2 \omega^* - \lambda^* \exp(\omega^* - \hat{u}) = \nabla^2 \hat{u} + \lambda^* e^{\omega^*}[1 - e^{-\hat{u}}], \qquad \text{in} \quad \Omega, \; t > 0, \tag{1.2.27}$$

$$\hat{u}(x,0) = \omega^*(x) - u_0(x) \qquad \text{in} \quad \Omega, \tag{1.2.28}$$

and

$$\frac{\partial \hat{u}}{\partial n} + \beta \hat{u} = 0 \qquad \text{on} \quad \partial\Omega, \; t > 0. \tag{1.2.29}$$

In this section we look for an upper solution to $\hat{u}$. To proceed, we estimate the term $(1 - e^{-\hat{u}})$ in (1.2.27) by noting that $0 \leq \hat{u} \leq \max[\omega^*, \omega^* - u_0]$ so that

$$e^{-\hat{u}} \geq 1 - \hat{u} + k\hat{u}^2 \tag{1.3.2}$$

for some k with $\frac{1}{2} > k > 0$. Hence

$$\hat{u}_t \leq \nabla^2 \hat{u} + \lambda^* e^{\omega^*}\{\hat{u} - k\hat{u}^2\} \qquad \text{in} \quad \Omega. \tag{1.3.3}$$

It follows that if there exists some $\psi_0(x,t)$ satisfying

$$\frac{\partial\psi_0}{\partial t} \geq \nabla^2\psi_0 + \lambda^* e^{\omega^*}(\psi_0 - k\psi_0^2) \qquad \text{in} \quad \Omega,\ t > 0, \tag{1.3.4}$$

$$\psi_0(x,0) \geq \hat{u}(x,0) - \omega^*(x) - u_0(x) \qquad \text{in} \quad \Omega, \tag{1.3.5}$$

and $$\frac{\partial\psi_0}{\partial n} + \beta\psi_0 \geq 0 \qquad \text{on} \quad \partial\Omega,\ t > 0, \tag{1.3.6}$$

then ψ_0 will be an upper solution to $\hat{u}$, and $u^* - \psi_0$ will be a lower solution to u^* (and hence also to u by virtue of (1.3.1)) provided inequality (1.3.2) holds.

Initially we are motivated by the asymptotic estimate of u in this region to choose ψ_0 such that

$$\psi_0(x,t) = \frac{\alpha_0\varphi^*}{t + t_0} + \frac{v_0(x)}{(t + t_0)^2} \tag{1.3.7}$$

for some constants α_0, t_0, and function $v_0(x)$ yet to be determined.

Again, φ^* is chosen as the solution to (1.2.3), (1.2.4) satisfying (1.2.34).

With this ψ_0, inequality (1.3.4) requires that

$$-\left\{\frac{\alpha_0\varphi^*}{(t+t_0)^2}+\frac{2v_0}{(t+t_0)^3}\right\}$$

$$\geq \frac{1}{(t+t_0)^2}\left\{\nabla^2 v_0+\lambda^* e^{\omega^*}v_0-k\lambda^* e^{\omega^*}\left[\alpha_0^2\varphi^{*2}+\frac{v_0^2}{(t+t_0)^2}+\frac{2\alpha_0\varphi^* v_0}{(t+t_0)}\right]\right\}$$

which is satisfied if

$$\nabla^2 v_0 + \lambda^* e^{\nu^*} v_0 \leq \alpha_0\varphi^*\left\{\alpha_0\varphi^* k\lambda^* e^{\nu^*}-1\right\} + \frac{2v_0}{(t+t_0)}\left\{\alpha_0\varphi^* k\lambda^* e^{\nu^*}-1\right\}. \tag{1.3.8}$$

We now choose v_0 such that

$$\nabla^2 v_0 + \lambda^* e^{\nu^*} v_0 = \alpha_0\varphi^*\{\alpha_0\varphi^* k\lambda^* e^{\omega^*}-1\} - d_0(x) \quad \text{in} \quad \Omega \tag{1.3.9}$$

with $\quad \dfrac{\partial v_0}{\partial n} + \beta v_0 = 0 \quad \text{on} \quad \partial\Omega \qquad (1.3.10)$

for some $d_0(x) \geq 0$ yet to be determined,

and with

$$v_0(x) \geq 0 \quad \text{throughout} \quad \Omega\,. \tag{1.3.11}$$

From the Fredholm Alternative, such a v_0 requires that

$$\int_\Omega (\varphi^*\{\alpha_0\varphi^*[\alpha_0\varphi^* k\lambda^* e^{\varphi^*} - 1]\} - \varphi^* d_0)\,dx = 0$$

and hence that

$$\int_\Omega \varphi^* d_0 = \alpha_0[\alpha_0 k\lambda^* I_3 - 1] \tag{1.3.12}$$

where again

$$I_3 = \int_\Omega \varphi^{*3} e^{\omega^*}.$$

Equation (1.3.12) is clearly satisfied, in light of (1.2.34), if we choose

$$d_0(x) = \alpha_0 \varphi^* [\alpha_0 k \lambda^* I_3 - 1] \quad \text{for} \quad x \in \Omega. \tag{1.3.13}$$

Now, with this v_0, inequality (1.3.8) becomes

$$d_0(x) \geq \frac{2\, v_0(x)}{(t+t_0)} \{1 - \alpha_0 \varphi^* k \lambda^* \varphi^* e^{\omega^*}\} \quad \text{for} \quad x \in \Omega, \quad t > 0. \tag{1.3.14}$$

In addition to inequality (1.3.14), ψ_0 will only be an upper solution to $\hat{u}$ if

both conditions (1.3.5) and (1.3.6) are also satisfied, i.e. if

$$\psi_0(x,0) = \frac{\alpha_0 \varphi^*(x)}{t_0} + \frac{v_0(x)}{t_0^2} \geq \omega^*(x) - u_0(x) \quad \text{for} \quad x \in \Omega, \tag{1.3.15}$$

and if $\dfrac{\partial \psi_0}{\partial n} + \beta \psi_0 \geq 0$ on $\partial\Omega$, $t > 0$. (1.3.16)

Condition (1.3.16) is now automatic by virtue of (1.2.4) and (1.3.10). Further, inequality (1.3.15) will also be satisfied if α_0 / t_0 is chosen sufficiently large.

It remains, therefore, to choose α_0 and t_0 to satisfy inequality (1.3.14).

Inequality (1.3.14) requires (as $v_0(x) \geq 0$) that

$$d_0(x) = \alpha_0 \varphi^* [\alpha_0 k \lambda^* I_3 - 1] > 0 \quad \text{for} \quad x \in \Omega, \tag{1.3.17}$$

and we must therefore choose α_0 to satisfy

$$\alpha_0 > [k\lambda^* I_3]^{-1}, \tag{1.3.18}$$

where I_3 is as defined in (1.2.58) and k as in (1.3.2).

Substituting for $d_0(x)$ by equation (1.3.13) in the inequality (1.3.14) indicates that α_0 and t_0 must be chosen such that

$$\alpha_0\varphi^*\{\alpha_0 k\lambda^* I_3 - 1\} \geq \frac{2v_0(x)}{(t+t_0)}\left\{1-\alpha_0\varphi^* k\lambda^* e^{\omega^*}\right\} \quad \text{for} \quad x\in\Omega, \quad t > 0. \tag{1.3.19}$$

From the requirement that α_0/t_0 be chosen large, (and because we wish to avoid choosing t_0 small to give us a chance at satisfying (1.3.19)) we see that α_0 must also be large.

If α_0 is 'large' then from equation (1.3.13) we have that

$$d_0(x) \sim \alpha_0\varphi^*[\alpha_0 k\lambda^* I_3 - 1] \tag{1.3.20}$$

which establishes that $d_0(x)$ is of order α_0^2.

Further, on substituting for $d_0(x)$ in equation (1.3.9), we find that

$$\nabla^2 v_0 + \lambda^* e^{\omega^*} v_0 \sim \alpha_0\varphi^*\left\{\alpha_0 k\lambda^*[\varphi^* e^{\omega^*} - I_3]\right\} \quad \text{in} \quad \Omega$$

with $\frac{\partial v_0}{\partial n} + \beta v_0 = 0$ on $\partial\Omega$

which suggests that, for 'large' α_0 , we may choose $v_0(x)$ to satisfy

$$v_0(x) \leq V_0\alpha_0^2\varphi^*(x) \quad \text{for} \quad x\in\Omega \tag{1.3.21}$$

for some positive order one constant V_0.

Inequality (1.3.19) requires that

$$\alpha_0\varphi^*\{\alpha_0 k\lambda^* I_3 - 1\} \geq \frac{2v_0(x)}{(t+t_0)}\{1 - \alpha_0\varphi^* k\lambda^* e^{\omega^*}\} \quad \text{for} \quad x\in\Omega, \quad t>0.$$

This is automatic, as $v_0(x) \geq 0,$ for all points $x\in\Omega$ at which

$$\alpha_0\varphi^* k\lambda^* e^{\omega^*} \geq 1,$$

and is otherwise satisfied, for all $t > 0,$ if

$$\alpha_0\varphi^*\{\alpha_0 k\lambda^* I_3 - 1\} \geq \frac{2v_0(x)}{t_0}, \quad x\in\Omega. \tag{1.3.22}$$

We consequently use (1.3.21) to estimate for $v_0(x)$ in (1.3.22) and see that the condition (1.3.19) will be satisfied for all $x\in\Omega,$ and $t > 0$ if

$$\alpha_0\{\alpha_0 k\lambda^* I_3 - 1\} \geq 2V_0\alpha_0^2/t_0 \tag{1.3.23}$$

and hence if t_0 is sufficiently large.

We conclude, therefore, that $\psi_0(x,t)$ as defined by (1.3.7) will be an upper solution to $\hat{u}$ for all $t > 0$ provided inequalities (1.3.15) and (1.3.23) hold which they will do if both α_0 / t_0 and t_0 are chosen suitably large.

To continue, condition (1.3.2) requires that

$$e^{-\hat{u}} \geq 1 - \hat{u} + k\hat{u}^2$$

for some $0 < k < ½$.

If we assume in addition that

$$\nabla^2 u_0 + \lambda^* e^{u_0} \geq 0 \quad \text{for} \quad x \in \Omega, \tag{1.3.24}$$

then the maximum principle applied to (1.2.23)-(1.2.25) yields that

$$u_t^*(x,t) \geq 0 \quad \text{for} \quad x \in \Omega, \quad t > 0$$

and hence that

$$\hat{u}_t = \frac{\partial}{\partial t}(\omega^* - u^*) = -u_t^* \leq 0 \quad \text{for} \quad x \in \Omega, \quad t > 0. \tag{1.3.25}$$

Condition (1.3.2) will be satisfied for all $\hat{u}$ if

$$\hat{u}(x,t) \leq \log\left(\frac{1}{2k}\right),$$

and from (1.3.25) we see that

$$\hat{u}(x,t) \le \hat{u}(x,0) = \omega^*(x) - u_0(x) \le \sup_{x\in\Omega}\{\omega^*(x) - u_0(x)\}$$

for all $x\in\Omega,\quad t > 0.$

Hence, condition (1.3.2) will be satisfied as required if we choose k such that $0 < k < \frac{1}{2}$ with

$$\sup_{x\in\Omega}\{\omega^* - u_0\} \le \log\left(\frac{1}{2k}\right), \tag{1.3.26}$$

and $\psi_0(x,t)$ is therefore an upper solution to $\hat{u}$ for all such k.

Having established an upper solution to $\hat{u}$ which exists for all $t > 0$ we now consider the problem of identifying a second upper solution to $\hat{u}$ which will exist for all sufficiently large t, but which does not require that α_0 be chosen large. If this second function is labelled $\psi_1(x,t)$ where

$$\psi_1(x,t) = \frac{\alpha_1 \varphi^*}{(t+t_1)} + \frac{v_1(x)}{(t+t_1)^2} \tag{1.3.27}$$

and if we maintain the assumption that $\hat{u}$ is restricted to allow use of the estimate (1.3.2), then following the analysis applied previously, we reach the conclusion that $\psi_1(x,t)$ will be an upper solution to $\hat{u}$ for all $t \geq \bar{t}$

provided

$$\alpha_1\varphi^*[\alpha_1 k\lambda^* I_3 - 1] \geq \frac{2v_1(x)}{(t+t_1)}\{1 - \alpha_1\varphi^* k\lambda^* e^{\omega^*}\} \quad \text{for all} \quad x\in\Omega, \quad t \geq \bar{t} \tag{1.3.28}$$

from (1.3.19), and if it can be established that

$$\psi_1(x,\bar{t}) = \frac{\alpha_1\varphi^*}{(\bar{t}+t_1)} + \frac{v_1(x)}{(\bar{t}+t_1)^2} \geq \hat{u}(x,\bar{t}) \quad \text{for} \quad x\in\Omega. \tag{1.3.29}$$

As we would now like to choose α_1 much closer to the minimum value

described by (1.3.18) we choose

$$\alpha_1 = [k\lambda^* I_3]^{-1} + \delta \tag{1.3.30}$$

for some $\delta > 0$ (and hopefully small) yet to be determined.

Then from (1.3.13) (with $\alpha_0 = \alpha_1$ and $d_0 = d_1$), we see that

$$d_1(x) = \alpha_1\varphi^*[\alpha_1 k\lambda^* I_3 - 1] = \delta\varphi^*[1+\delta(k\lambda^* I_3)]. \tag{1.3.31}$$

From equations (1.3.9), (1.3.10), (with $v_0 = v_1, d_0 = d_1$) and on substituting d_1

by (1.3.31), then

$$\nabla^2 v_1 + \lambda^* e^{\omega^*} v_1 = [(k\lambda^* I_3)^{-1} + \delta]^2 \varphi^* k\lambda^* [\varphi^* e^{\omega^*} - I_3] \quad \text{in} \quad \Omega$$

with $\frac{\partial v_1}{\partial n} + \beta v_1 = 0$ on $\partial\Omega$

which suggests that we may choose v_1 to satisfy

$$v_1(x) \le V_1\varphi^*(x) \quad \text{for} \quad x\epsilon\Omega \tag{1.3.32}$$

and for some positive constant V_1 .

To establish ψ_1 as an upper solution to $\hat{u}$ we must satisfy condition (1.3.28).

As $v_1(x) \ge 0,$ however, inequality (1.3.28) will be automatic if

$$\alpha_1\varphi^* k\lambda^* e^{\omega^*} \ge 1$$

and will otherwise be satisfied for all $t \ge \overline{t}$ (on substitiuting for $v_1(x)$ by (1.3.32)) if

$$\alpha_1[\alpha_1 k\lambda^* I_3 - 1] \ge \frac{2V_1}{(\overline{t}+t_1)}$$

i.e. if $$\delta[1+\delta(k\lambda * I_3)] \ge \frac{2V_1}{(\overline{t}+t_1)} \tag{1.3.33}$$

We must also ensure satisfaction of inequality (1.3.32), i.e. that

$$\psi_1(x,\overline{t}) = \frac{\alpha_1\varphi^*}{(\overline{t}+t_1)} + \frac{v_1(x)}{(\overline{t}+t_1)^2} \ge \hat{u}(x,\overline{t}) .$$

To to this we shall make use of our information on ψ_0 .

We have previously established that $\psi_0(x,t)$ is an upper solution to $\hat{u}$ for all

$t > 0$ (and hence at $t = \bar{t}$) provided α_0 and t_0 are chosen suitably. It

follows that if

$$\psi_1(x,\bar{t}) \geq \psi_0(x,\bar{t}) \quad \text{for} \quad x\epsilon\Omega$$

ie. if $$\frac{\alpha_1\varphi^*}{(\bar{t}+t_1)} + \frac{v_1(x)}{(\bar{t}+t_1)^2} \geq \frac{\alpha_0\varphi^*}{(\bar{t}+t_0)} + \frac{v_0(x)}{(\bar{t}+t_0)^2} \quad \text{for} \quad x\epsilon\Omega, \tag{1.3.34}$$

then condition (1.3.29) will hold as required.

Hence, as $v_1(x) \geq 0,$ and $v_0(x)$ satisfies the estimate (1.3.21), inequality

(1.3.34) will hold if

$$\frac{\alpha_1}{(\bar{t}+t_1)} \geq \frac{\alpha_0}{(\bar{t}+t_0)} + \frac{V_0\alpha_0^2}{(\bar{t}+t_0)^2}. \tag{1.3.35}$$

We assume, therefore, that

$$\bar{t} \geq V_0\alpha_0 \tag{1.3.36}$$

in which case

$$\frac{V_0\alpha_0^2}{(\bar{t}+t_0)^2} < \frac{\alpha_0}{(\bar{t}+t_0)}$$

and inequality (1.3.35) is satisfied if

$$\frac{\alpha_1}{(\bar{t}+t_1)} \geq \frac{2\alpha_0}{(\bar{t}+t_0)}$$

i.e. if

$$(\bar{t}+t_1) \le \frac{\alpha_1(\bar{t}+t_0)}{2\alpha_0}. \tag{1.3.37}$$

We conclude that $\psi_1(x,t)$ will be an upper solution to $\hat{u}$ for $t \ge \bar{t}$

provided both inequalities (1.3.33) and (1.3.37) are satisfied.

To ensure satisfaction of condition (1.3.37) we choose

$$t_1 = -\left[1 - \frac{\alpha_1}{2\alpha_0}\right]\bar{t} \tag{1.3.38}$$

in which case (1.3.33) requires that

$$\frac{\alpha_1\bar{t}}{2\alpha_0} \ge \frac{V_1}{\delta[1+\delta\,(k\lambda * I_3)]}$$

i.e.

$$\bar{t} \ge \frac{2\alpha_0 V_1}{\alpha_1\delta[1+\delta(k\lambda * I_3)]}. \tag{1.3.39}$$

Hence, if α_0' is defined as

$$\alpha_0' = \frac{2\alpha_0 V_1}{\alpha_1[1+\delta\,(k\lambda * I_3)\,]} = C\alpha_0, \tag{1.3.40}$$

then the condition (1.3.39) requires that

$$\bar{t} \geq \frac{\alpha_0'}{\delta}.$$

With α_0' thus defined, equation (1.3.38) and inequality (1.3.39) reduce to

$$t_1 = -\left\{1 - \frac{\alpha_1 C}{2\alpha_0'}\right\}\bar{t} \quad \text{with} \quad \bar{t} \geq \frac{\alpha_0'}{\delta} \tag{1.3.41}$$

and the condition (1.3.36) is satisfied as required.

It remains, finally, to identify k. We see that ψ_1 will only be an upper solution to $\hat{u}$ for $t \geq \bar{t}$ if the estimate (1.3.2) is valid for $t \geq \bar{t}$, i.e. if

$$e^{-\hat{u}} \geq 1 - \hat{u} + k\hat{u}^2 \quad \text{for} \quad x \in \Omega, \; t \geq \bar{t}$$

with $0 < k < ½$.

Again, however, we find that this estimate is valid if

$$\hat{u}(x,t) \leq \log\left(\frac{1}{2k}\right) . \tag{1.3.42}$$

Having established that $\psi_0(x,t)$ is an upper solution to $\hat{u}$ for all $t > 0$, we are now able to derive a less restrictive condition on k for $t \geq \bar{t}$ than that required in (1.3.26).

As ψ_0 is an upper solution to $\hat{u}$, we see that condition (1.3.42) will hold as required for all $t \geq \bar{t}$ if

$$\psi_0(x,\bar{t}) = \frac{\alpha_0\varphi^*}{(\bar{t}+t_0)} + \frac{v_0(x)}{(\bar{t}+t_0)^2} \leq \log\left(\frac{1}{2k}\right). \tag{1.3.43}$$

If we estimate $v_0(x)$ by (1.3.21) and as (1.3.36) remains valid, we see that inequality (1.3.43) is itself satisfied if

$$\log\left(\frac{1}{2k}\right) \geq \frac{2\alpha_0\varphi^*}{(\bar{t}+t_0)}$$

which, as $t_0 \geq 0$, will hold if

$$\log\left(\frac{1}{2k}\right) \geq \frac{M\alpha_0}{\bar{t}} - \frac{M\alpha_0'}{C\bar{t}} \tag{1.3.44}$$

by (1.3.40) with $M = 2\sup_{x\in\Omega}\{\varphi^*(x)\}$.

Hence,

$$\psi_1(x,t) = \frac{\alpha_1\varphi^*}{(t+t_1)} + \frac{v_1(x)}{(t+t_1)^2}$$

is an upper solution to $\hat{u} = \omega^* - u^*$ where for large α_0 and small $\delta > 0$,

$$\bar{t} \geq \frac{\alpha_0'}{\delta}, \quad t_1 = -\left[1 - \frac{\alpha_1 C}{2\alpha_0'}\right]\bar{t}, \quad \text{and} \quad \alpha_1 = [\,k\lambda^* I_3\,]^{-1} + \delta \;.$$

Further, $\alpha_0' = C\alpha_0$ with

$$C^{-1} = \frac{\alpha_1[1+\delta(k\lambda^* I_3)]}{2V_1}$$

for positive constant V_1 and

$$\log\left(\frac{1}{2k}\right) \geq \frac{\mathrm{M}\alpha_0'}{C\bar{t}}$$

where $\mathrm{M} = 2\sup_{x\in\Omega}\{\varphi^*(x)\}$.

It follows as

$$\hat{u}(x,t) = \omega^*(x) - u^*(x,t) \leq \psi_1(x,t) \quad \text{for} \quad x\in\Omega,\, t \geq \bar{t}$$

and as $u(x,t) \geq u^*(x,t)$ by (1.3.1), that

$$u(x,t) \geq u^*(x,t) \geq \omega^* - \psi_1(x,t) \quad \text{for} \quad x\in\Omega,\, t \geq \bar{t}$$

and hence that

$$u(x,t) \geq \omega^*(x) - \frac{\alpha_0\varphi^*(x)}{(t+t_1)} - \frac{V_1(x)}{(t+t_1)^2} \quad \text{for} \quad x\in\Omega, \quad t \geq \bar{t} \tag{1.3.45}$$

Inequality (1.3.45) allows us to estimate $u(x,t)$ from below for any $t \geq \bar{t}$.

We now investigate this estimate for any such t.

We first choose equality in condition (1.3.44) so that

$$2k = \exp\left[-\frac{M\alpha_0'}{C\bar{t}}\right]. \tag{1.3.46}$$

Hence, as $\alpha_0'/\bar{t}$ is 'large', it follows that

$$2k \geq 1 - \frac{B\alpha_0'}{\bar{t}} \geq 1 - B\delta$$

for some positive constant B.

At $t \geq \bar{t}$, inequality (1.3.45) yields

$$\begin{aligned} u(x,t) &\geq \omega^* - \frac{\alpha_1\varphi^*}{(t+t_1)} - \frac{v_1(x)}{(t+t_1)^2} \\ &= \omega^* - \frac{\left[(k\lambda^* I_3)^{-1} + \delta\right]\varphi^*}{\left[t - \left(1 - \dfrac{\alpha_1 C}{2\alpha_0'}\right)\bar{t}\right]} - \frac{v_1(x)}{\left[t - \left(1 - \dfrac{\alpha_1 C}{2\alpha_0'}\right)\bar{t}\right]^2}. \end{aligned} \tag{1.3.47}$$

Now, as both $v_1(x)$ and $\dfrac{\alpha_1 C}{2\alpha_0'}$ are strictly positive, we see from (1.3.47) that

$$u(x,t) \geq \omega^* - \frac{[(k\lambda^* I_3)^{-1} + \delta]\varphi^*}{(t-\bar{t})} - \frac{v_1(x)}{(t-\bar{t})^2} \quad \text{for} \quad x \in \Omega, \ t \geq \bar{t}. \tag{1.3.48}$$

Further, with k as in equation (1.3.46)

$$[k\lambda^* I_3]^{-1} \leq [\lambda^* I_3]^{-1}[\tfrac{1}{2} - \tfrac{1}{2}B\delta]^{-1} \leq [\tfrac{1}{2}\lambda^* I_3]^{-1} + B_2\delta \tag{1.3.49}$$

for positive B_2 choosing δ 'small'.

Applying the estimate (1.3.49) in (1.3.48) then establishes that

$$u(x,t) \geq \omega^* - \frac{\varphi^*}{(\tfrac{1}{2}\lambda^* I_3)(t-\bar{t})} - \frac{B_2\delta\varphi^*}{(t-\bar{t})} - \frac{v_1(x)}{(t-\bar{t})^2} \quad \text{for} \quad x \in \Omega, \ t \geq \bar{t}.$$

Finally, as $v_1(x) \geq 0$ and satisfies (1.3.32), we see that there exists a positive

V_1 such that

$$u(x,t) \geq \omega^* - \frac{\varphi^*}{(\tfrac{1}{2}\lambda^* I_3)(t-\bar{t})} - \frac{B_2\delta\varphi^*}{(t-\bar{t})} - \frac{V_1\varphi^*}{(t-\bar{t})^2} \quad \text{for} \quad x \in \Omega, \ t \geq \bar{t}. \tag{1.3.50}$$

We next assume, in addition to $t \geq \bar{t} \geq \alpha_0'/\delta$, that

$$\bar{t}/t \text{ is 'small'}. \tag{1.3.51}$$

If this is the case, then there exists a positive constant, B_3 say, such that

$$\frac{1}{(t-\bar{t})} = \frac{1}{t[1-\bar{t}/t]} \leq \frac{1}{t} + \frac{B_3\bar{t}}{t^2}.$$

Applying this estimate in (1.3.50) then shows that

$$\begin{aligned} u(x,t) \geq \omega^* - \frac{\varphi^*}{(\tfrac{1}{2}\lambda^* I_3 t)} - \frac{\varphi^* B_3 \bar{t}}{(\tfrac{1}{2}\lambda^* I_3) t^2} \\ - \frac{B_2 \delta \varphi^*}{t} - \frac{B_2 B_3 \delta \varphi^* \bar{t}}{t^2} \\ - V_1 \varphi^* \left[\frac{1}{t^2} + \frac{2B_3 \bar{t}}{t^3} + \frac{B_3^2 \bar{t}^2}{t^4} \right] \end{aligned} \tag{1.3.52}$$

provided $t \geq \bar{t} \geq \alpha_0' / \delta$ and $\bar{t}/t$ is 'small'.

1.3.2 A lower solution in time region II

Section 1.3.1 has established a lower solution to u which exists for all sufficiently large times but which, by definition, never exceeds ω^*. Further, the analysis of Section 1.2.2 has also found an upper solution to u in this time region which shows that $u<\omega^*$ at least until

$$t \le t_1(\lambda-\lambda^*)^{-1/2} .$$

The results of these upper and lower solutions may be combined, and using (1.2.49) and (1.3.45) yield that, if $t = c(\lambda-\lambda^*)^{-1/2}$ with $c<t_1$, then there exists constants C_1 and C_2 such that

$$\omega^* - C_1\varphi^*(\lambda-\lambda^*)^{1/2} \le u(x,t) \le \omega^* - C_2\varphi^*(\lambda-\lambda^*)^{1/2}. \tag{1.3.53}$$

In light of this analysis, we now look for a second lower solution to u which applies for t of the order of $(\lambda-\lambda^*)^{-1/2}$ and which exhibits the form of u identified. If we call this function $\underline{u}(x,t)$, then $\underline{u}(x,t)$ will be a lower solution to u for all $t \ge \underline{t}_0(=\underline{T}_0(\lambda-\lambda^*)^{-1/2})$ say, if

$$\underline{u}_t \le \nabla^2\underline{u} + \lambda e^{\underline{u}} \qquad \text{in } \Omega,\ t \ge \underline{t}_0, \tag{1.3.54}$$

$$\underline{u}(x,\underline{t}_0) \le u(x,\underline{t}_0) \qquad \text{in} \quad \Omega, \tag{1.3.55}$$

and
$$\frac{\partial \underline{u}}{\partial n} + \beta \underline{u} \le 0 \qquad \text{on} \quad \partial\Omega,\, t \ge \underline{t}_0. \tag{1.3.56}$$

Motivated by the estimates (1.3.53) and by the asymptotic analysis of Lacey 1983, we set

$$\epsilon = (\lambda-\lambda^*) \tag{1.3.57}$$

which we assume to be 'small', and look for $\underline{u}(x,t)$ such that

$$\underline{u}(x,t) = \omega^*(x) + \epsilon^{1/2}\underline{u}_1(x,t) + \epsilon\underline{u}_2(x,t) \tag{1.3.58}$$

where $\underline{u}_1$ and $\underline{u}_2$ are some functions yet to be determined.

With this $\underline{u}$, inequality (1.3.54) requires that

$$\epsilon^{1/2}\underline{u}_{1t} + \epsilon\underline{u}_{2t} - [\nabla^2\omega^* + \epsilon^{1/2}\nabla^2\underline{u}_1 + \epsilon\nabla^2\underline{u}_2] - \lambda e^{\omega^*}\exp[\epsilon^{1/2}\underline{u}_1 + \epsilon\underline{u}_2] \le 0$$
$$\text{for} \quad x\in\Omega,\, t \ge \underline{t}_0. \tag{1.3.59}$$

We next rescale t in terms of the order one variable τ defined as

$$\tau = (\lambda-\lambda^*)^{1/2}t = \epsilon^{1/2}t \tag{1.3.60}$$

in which case inequality (1.3.59) becomes

$$\epsilon\underline{u}_{1\tau} + \epsilon^{3/2}\underline{u}_{2\tau} - (\nabla^2\omega^* + \epsilon^{1/2}\nabla^2\underline{u}_1 + \epsilon\nabla^2\underline{u}_2) - (\lambda^* + \epsilon)e^{\omega^*}\exp(\epsilon^{1/2}\underline{u}_1 + \epsilon\underline{u}_2) \le 0$$

$$\text{for} \quad x\in\Omega,\, t \ge \underline{t}_0 \tag{1.3.61}$$

Now, the order one terms in (1.3.61) are

$$\nabla^2 \omega^* + \lambda^* e^{\omega^*} = 0$$

in light of (1.2.1).

The terms in $\epsilon^{1/2}$, i.e.

$$\nabla^2 u_1 + \lambda^* e^{\omega^*} u_1$$

will also cancel if we choose

$$u_1(x,\tau) = a(\tau)\varphi^*(x) \tag{1.3.62}$$

for φ^* satisfying (1.2.3), (1.2.4) and (1.2.34) and any function $a(\tau)$.

The largest remaining terms in (1.3.61) are now of order ϵ. To ensure satisfaction of inequality (1.3.61) at this level we require that

$$u_{1\tau} - [\nabla^2 u_2 + e^{\omega^*} + \lambda^* e^{\omega^*}(u_2 + \tfrac{1}{2}u_1^2)] \le 0 \tag{1.3.63}$$

which, on substituting for u_1 by (1.3.62), becomes

$$a_\tau \varphi^* - (\nabla^2 u_2 + e^{\omega^*} + \lambda^* e^{\omega^*} u_2 + \lambda^* e^{\omega^*} a^2 \varphi^{*2}) \le 0 \quad \text{for} \quad x \in \Omega, t \ge t_0. \tag{1.3.64}$$

On rearranging inequality (1.3.64) we find that we require

$$\nabla^2 u_2 + \lambda^* e^{\omega^*} u_2 \ge a_\tau \varphi^* - (e^{\omega^*} + \tfrac{1}{2}\lambda^* e^{\omega^*} a^2 \varphi^{*2})$$

which, on multiplying through by φ^* and integrating over Ω, yields that $a(\tau)$

must be chosen to satisfy

$$a_\tau \leq I_1 + \tfrac{1}{2}\lambda^* I_3 a^2 \tag{1.3.65}$$

where

$$I_1 = \int_\Omega \varphi^* e^{\omega^*} \quad \text{and} \quad I_3 = \int_\Omega \varphi^{*3} e^{\omega^*}.$$

Inequality (1.3.65) in turn yields

$$a(\tau) \leq \left[\frac{I_1}{\tfrac{1}{2}\lambda^* I_3}\right]^{\frac{1}{2}} \tan\left[(\tfrac{1}{2}\lambda^* I_1 I_3)^{\frac{1}{2}}\tau - c\right]$$

for some constant c.

From our lower solution for u in Section 1.3.1, however,

$$u \geq \omega^* - \frac{\varphi^*}{\tfrac{1}{2}\lambda^* I_3 t} \quad \text{for} \quad t \sim \epsilon^{-\frac{1}{2}}$$

and $a(\tau)$ can be chosen to be consistent with this estimate if we set $c = \pi/2$. We consequently choose

$$a(\tau) = C_1 \tan[C_2 \tau - \pi/2] \tag{1.3.66}$$

for constants C_1 and C_2 chosen to ensure satisfaction of inequality (1.3.65). This requires that

$$a_\tau = C_1 C_2 + C_2/C_1 a^2 \leq I_1 + \tfrac{1}{2}\lambda^* I_3 a^2 \tag{1.3.67}$$

and to hold for all possible a we set

$$\mathcal{C}_1\mathcal{C}_2 \le I_1 \tag{1.3.68}$$

and $$\mathcal{C}_2/\mathcal{C}_1 \le \tfrac{1}{2}\lambda^* I_3 . \tag{1.3.69}$$

With this choice of $a(\tau)$ inequality (1.3.64) requires that

$$\begin{aligned}\nabla^2 u_2 + \lambda^* e^{\omega^*} u_2 \ge\ & a_\tau \varphi^* - \left[e^{\omega^*} + \tfrac{1}{2}\lambda^* e^{\omega^*} a^2 \varphi^{*2}\right] \\ & = \left[\mathcal{C}_1\mathcal{C}_2\varphi^* - e^{\omega^*}\right] + \left[\mathcal{C}_2/\mathcal{C}_1 - \tfrac{1}{2}\lambda^* e^{\omega^*}\varphi^*\right]\varphi^* a^2 \\ & \text{for } x\in\Omega \ \ t \ge t_0 .\end{aligned} \tag{1.3.70}$$

We now choose $u_2(x,t)$ such that

$$\begin{aligned}\nabla^2 u_2 + \lambda^* e^{\omega^*} u_2 = \left[\mathcal{C}_1\mathcal{C}_2\varphi^* - e^{\omega^*}\right] + \left[\mathcal{C}_2/\mathcal{C}_1 - \tfrac{1}{2}\lambda^* e^{\omega^*}\varphi^*\right]\varphi^* a^2 + d(x,\tau) \\ \text{for } x\in\Omega, \tau \ge \tau_0\end{aligned} \tag{1.3.71}$$

with $$\frac{\partial u_2}{\partial n} + \beta u_2 = 0 \quad \text{on} \quad \partial\Omega,\ \tau \ge \tau_0 \tag{1.3.72}$$

for some function $d(x,\tau) \ge 0$ yet to be determined.

On multiplying both sides of equation (1.3.71) by φ^* and integrating over Ω we

find that the existence of such a u_2 requires

$$(\mathcal{C}_1\mathcal{C}_2 - I_1) + (\mathcal{C}_2/\mathcal{C}_1 - \tfrac{1}{2}\lambda^* I_3)a^2 + \int_\Omega \varphi^* d = 0 \quad \text{for all} \quad \tau \ge \tau_0 . \tag{1.3.73}$$

Equation (1.3.73) is clearly satisfied in light of (1.2.34) if

$$d(x,\tau) = \varphi^* d(\tau), \tag{1.3.74}$$

and

$$d(\tau) = (I_1 - \underline{C}_1\underline{C}_2) + (\tfrac{1}{2}\lambda^* I_3 - \underline{C}_2/\underline{C}_1)\underline{a}^2(\tau). \tag{1.3.75}$$

With $\underline{u}_2$ chosen to satisfy (1.3.71) and $d(x,\tau)$ as defined by (1.3.74), (1.3.75), we see that the order ϵ terms in inequality (1.3.61) (as given by the left hand side of (1.3.63)) are

$$-\epsilon d(\tau)\varphi^*$$

and are less than or equal to zero as required.

Returning to the 'full' inequality (1.3.61) and substituting for $\underline{u}_1(x,t)$ by (1.3.62) we find that one condition required for $\underline{u}$ to be a lower solution to u for $t \geq \underline{t}_0$ is that

$$\begin{aligned} &-\epsilon d(\tau)\varphi^* + \epsilon^{3/2}\underline{u}_{2\tau} \\ &\qquad - \lambda^* e^{\omega^*}\left\{\exp\left(\epsilon^{\frac{1}{2}}\underline{a}\varphi^* + \epsilon\underline{u}_2\right) - \left(1 + \epsilon^{\frac{1}{2}}\underline{a}\varphi^* + \tfrac{1}{2}\epsilon\underline{a}^2\varphi^{*2} + \epsilon\underline{u}_2\right)\right\} \\ &\qquad\qquad - \epsilon e^{\omega^*}\left\{\exp\left(\epsilon^{\frac{1}{2}}\underline{a}\varphi^* + \epsilon\underline{u}_2\right)\right\} \leq 0 \\ &\qquad\qquad\qquad \text{for } x\in\Omega,\ \tau \geq \underline{\tau}_0. \end{aligned} \tag{1.3.76}$$

Following some manipulation, it can be shown that inequality (1.3.76) is equivalent to

$$-\epsilon d(\tau)\varphi^* + \epsilon^{3/2}u_{2\tau} - \lambda^* e^{\omega^*}\{\exp(\epsilon^{1/2}\underline{a}\varphi^*) - (1+\epsilon^{1/2}\underline{a}\varphi^* + \tfrac{1}{2}\epsilon\underline{a}^2\varphi^{*2})$$

$$+ \left[\exp(\epsilon^{1/2}\underline{a}\varphi^*) - 1\right]\left[\exp(\epsilon\underline{u}_2)-1\right]$$

$$+ \exp(\epsilon\underline{u}_2) - (1 + \epsilon\underline{u}_2)\}$$

$$-\epsilon e^{\omega^*}\left\{\exp\left(\epsilon^{1/2}\underline{u}\varphi^* + \epsilon\underline{u}_2\right) - 1\right\} \le 0. \quad (1.3.77)$$

If we define the term E by

$$E = \exp(\epsilon^{1/2}\underline{a}\varphi^*) - (1 + \epsilon^{1/2}\underline{a}\varphi^* + \tfrac{1}{2}\epsilon\underline{a}^2\varphi^{*2}),$$

then clearly

$$E \ge \frac{\epsilon^{3/2}\,\underline{a}^3\varphi^{*3}}{6} \qquad \text{for all } \underline{a}(\tau). \quad (1.3.78)$$

We next consider a term F defined as

$$F = (e^x - 1)(e^y - 1) \qquad \text{for any } x, y.$$

Clearly, if x and y are of the same sign, then

$$F \ge 0.$$

Alternatively, if $x \le 0,$ then

$$(e^x - 1) \ge -(e^{|x|} - 1) \quad (1.3.79)$$

and we see that, if x and y are of opposite signs then

$$F \ge -(e^{|x|} - 1)(e^{|y|} - 1).$$

Applying this estimate with $x = \epsilon^{1/2}\underline{a}\varphi^*$ and $y = \epsilon\underline{u}_2$ yields that

$$[\exp(\epsilon^{1/2}\underline{a}\varphi^*) - 1]\,[\exp(\epsilon\underline{u}_2) - 1]$$
$$\geq -[\exp(\epsilon^{1/2}\,|\,\underline{a}\,|\,\varphi^*) - 1]\,[\exp(\epsilon\,|\,\underline{u}_2\,|\,) - 1] \qquad \text{for all } x,\tau. \tag{1.3.80}$$

Continuing in this vein, and applying the estimate (1.3.79) with $x = y + z,$ we see that

$$e^{(y+x)} - 1 \geq 0 \quad \text{if} \quad (y+z) \geq 0$$

and

$$e^{(y+z)} - 1 \geq -\{e^{|y+z|} - 1\}$$
$$\geq -\{e^{(|y|+|z|)} - 1\} \qquad \text{otherwise.}$$

Taking $y = \epsilon^{1/2}\underline{a}\varphi^*$ and $z = \epsilon\underline{u}_2$ yields

$$\exp\left\{\epsilon^{1/2}\underline{a}\varphi^* + \epsilon\underline{u}_2\right\} \geq -\left\{\exp\left(\epsilon^{1/2}\,|\,\underline{a}\,|\,\varphi^* + \epsilon\,|\,\underline{u}_2\,|\,\right) - 1\right\} \qquad \text{for all } x,\tau. \tag{1.3.81}$$

Finally, we have for any y, that

$$e^{y} - (1 + y) \geq 0$$

(from Taylor's Theorem), so that

$$\exp(\epsilon\underline{u}_2) - (1+\epsilon\underline{u}_2) \geq 0 \quad \text{for all } x,\tau. \tag{1.3.82}$$

Each of the inequalities (1.3.78), (1.3.80), (1.3.81) and (1.3.82) allows us to estimate for a term in the left hand side of inequality (1.3.77) so that it suffices to show that

$$- \epsilon d(\tau)\varphi^* + \epsilon^{3/2}u_{2\tau}$$

$$+ \lambda^* e^{\omega^*}\left\{ - \frac{\epsilon^{3/2}a^3\varphi^{*3}}{6} + [\exp(\epsilon^{1/4}|a|\varphi^*) - 1]\,[\exp(\epsilon|u_2|) - 1]\right\}$$

$$+ \epsilon e^{\omega^*}\{\exp(\epsilon^{1/4}|a|\varphi^* + \epsilon|u_2|) - 1\} \le 0 \qquad (1.3.83)$$

We next return to equations (1.3.71) and (1.3.72) in order to estimate the 'size' of u_2.

On substituting for $a(x,\tau)$ by equations (1.3.74), (1.3.75) in (1.3.71) we find that $u_2(x,\tau)$ satisfies

$$\nabla^2 u_2 + \lambda^* e^{\omega^*} u_2 = \left[I_1\varphi^* - e^{\omega^*}\right] + \tfrac{1}{2}\lambda^*\varphi^* a^2\,[I_3 - e^{\omega^*}\varphi^*] \quad \text{in} \quad \Omega, \tau \ge \tau_0$$

with $$\frac{\partial u_2}{\partial n} + \beta u_2 = 0 \quad \text{on} \quad \partial\Omega, \tau \ge \tau_0,$$

and that $u_2(x,\tau)$ may be chosen such that

$$|u_2(x,\tau)| \le E_1\varphi^*\,[1 + a^2] \quad \text{for} \quad x\in\Omega, \tau \ge \tau_0, \qquad (1.3.84)$$

for some positive constant E_1.

Further, on differentiating with respect to τ and using equation (1.3.67) to substitute for a_τ, we find that

$$\nabla^2 u_{2\tau} + \lambda^* e^{\omega^*} u_{2\tau} = \lambda^*\varphi^* a\,[I_3 - e^{\omega^*}\varphi^*]\,[C_1C_2 + C_2/C_1 a^2] \quad \text{in} \quad \Omega, \tau \ge \tau_0$$

with $\dfrac{\partial \underline{u}_{2\tau}}{\partial n} + \beta \underline{u}_{2\tau} = 0$ on $\partial\Omega, \tau \geq \underline{\tau}_0$.

Hence, $\underline{u}_2$ may also be chosen such that

$$|\underline{u}_{2\tau}(x,\tau)| \leq E_2\varphi^* \, [1 + |\underline{a}|^3] \quad \text{for} \quad x\in\Omega, \tau \geq \underline{\tau}_0, \tag{1.3.85}$$

and some positive constant E_2.

We now return to inequality (1.3.83) and see that if

$$\epsilon^{1/2}|\underline{a}|\varphi^* < 1 \quad \text{and} \quad \epsilon|\underline{u}_2| < 1, \tag{1.3.86}$$

then there exist positive constants M_1 and M_2 for which

$$\begin{aligned} &\exp(\epsilon^{1/2}|\underline{a}|\varphi^*) \leq 1 + M_1\epsilon^{1/2}|\underline{a}|\varphi^* \\ \text{and} \\ &\exp(\epsilon|\underline{u}_2|) \leq 1 + M_2\epsilon|\underline{u}_2|, \end{aligned} \tag{1.3.87}$$

say for example

$$\begin{aligned} &M_1 = 1 + \epsilon^{1/2}|\underline{a}|\varphi^*\exp(\epsilon^{1/2}|\underline{u}|\varphi^*) \\ \text{and} \\ &M_2 = 1 + \epsilon|\underline{u}_2|\exp(\epsilon|\underline{u}_2|). \end{aligned} \tag{1.3.88}$$

Hence, if (1.3.86) holds, then inequality (1.3.83) will be satisfied if

$$\begin{aligned} &-\epsilon d(\tau)\varphi^* + \epsilon^{3/2}\underline{u}_{2\tau} \\ &\qquad + \lambda^* e^{\omega^*}\left\{-\frac{\epsilon^{3/2}\underline{a}^3\varphi^{*3}}{6} + (M_1\epsilon^{1/2}|\underline{a}|\varphi^*)(M_2\epsilon|\underline{u}_2|)\right\} \\ &\qquad\qquad + \epsilon e^{\omega^*}\left\{(1 + M_1\epsilon^{1/2}|\underline{a}|\varphi^*)(1 + M_2\epsilon|\underline{u}_2|) - 1\right\} \leq 0 \end{aligned}$$

i.e. if

$$-\epsilon d(\tau)\varphi^* + \epsilon^{3/2} u_{2\tau}$$

$$+\lambda^* e^{\omega *}\left\{-\frac{\epsilon^{3/2} a^3 \varphi^{*3}}{6} + M_1 M_2 \epsilon^{3/2} |a||u_2|\varphi^*\right\}$$

$$+ \epsilon e^{\omega^*}\{M_1 \epsilon^{1/2} |a|\varphi^* + M_2 \epsilon |u_2| + \epsilon^{3/2} M_1 M_2 |a||u_2|\varphi^*\} \le 0 \tag{1.3.89}$$

In light of the estimate (1.3.84) we see that (1.3.86) will be satisfied as required if we assume that

$$|a(\tau)| \le M_4 \epsilon^{-1/2} \tag{1.3.90}$$

for some suitable positive constant M_4.

Hence, on substituting for u_2 and $u_{2\tau}$ using the estimates (1.3.84) and (1.3.85) in (1.3.89) we arrive at an inequality which we seek to satisfy in order that $u(x,\tau)$ may be a lower solution to u for all $\tau \ge \tau_0$, that is

$$\epsilon^{3/2} E_2 \varphi^* [1 + |a|^3]$$

$$+ \lambda^* e^{\omega^*}\left\{\frac{-\epsilon^{3/2} a^3 \varphi^{*3}}{6} + M_1 M_2 \epsilon^{3/2} e_1 \varphi^{*2} |a| [1 + a^2]\right\}$$

$$+ \epsilon e^{\omega^*}\{M_1 \epsilon^{1/2} |a|\varphi^* + M_2 \epsilon E_1 \varphi^* [1 + a^2]$$

$$+ \epsilon^{3/2} M_1 M_2 E_1 \varphi^{*2} |a| [1 + a^2]\} \le \epsilon d(\tau)\varphi^*$$

$$x \in \Omega, \tau \ge \tau_0 . \tag{1.3.91}$$

It remains to try to explicitly define a function $a(\tau)$ which will ensure satisfaction of this inequality, i.e. we must choose the constants C_1 and C_2 to satisfy the estimates (1.3.68) and (1.3.69).

Before doing this, we first recognise that satisfaction of inequality (1.3.91) alone will not allow the conclusion that $\underline{u}$ is a lower solution to u ; we must also ensure that our chosen $\underline{u}$ satisfies the necessary initial and boundary conditions (1.3.55) and (1.3.56). In addition, if a range of lower solutions is available, we would clearly like to choose that which will become 'large' in the 'shortest' time, and which will consequently allow the best upper estimate to the blow up time t_b of u.

The boundary condition (1.3.56) requires that

$$\frac{\partial \underline{u}}{\partial n} + \beta \underline{u} \le 0 \quad \text{on} \quad \partial\Omega, \tau \ge \underline{\tau}_0.$$

and is automatic in light of our choice of functions $\underline{u}_1$ and $\underline{u}_2$, , i.e. from (1.2.4) and (1.3.72).

Next, before completely specifying $\underline{C}_1$ and $\underline{C}_2$, we investigate the consequences of any choice made on both the other factors which may influence this decision, i.e. the size of $\underline{u}(x, \underline{\tau}_0)$ and the resulting blow up time of the function $\underline{u}$.

The constants $\underline{C}_1$ and $\underline{C}_2$ <u>must</u> be chosen to satisfy the inequalities (1.3.68) and (1.3.69), i.e.

$$\underline{C}_1\underline{C}_2 \le I_1 \quad \text{and} \quad \underline{C}_2/\underline{C}_1 \le \tfrac{1}{2}\lambda^* I_3.$$

We therefore assume that

$$\underline{C}_1\underline{C}_2 = I_1-\underline{k}_1 \quad \text{and} \quad \underline{C}_2/\underline{C}_1 = \tfrac{1}{2}\lambda^* I_3-\underline{k}_2 \tag{1.3.92}$$

for some positive constants $\underline{k}_1$ and $\underline{k}_2$.

Hence,

$$\underline{C}_1 = \left\{\frac{I_1-\underline{k}_1}{\tfrac{1}{2}\lambda^* I_3-\underline{k}_2}\right\}^{1/2} \tag{1.3.93}$$

and $$\underline{C}_2 = \left\{(\tfrac{1}{2}\lambda^* I_3-\underline{k}_2)\ (I_1-\underline{k}_1)\right\}^{1/2}. \tag{1.3.94}$$

Now from equation (1.3.66)

$$\underline{a}(\tau) = \underline{C}_1 \tan[\underline{C}_2\tau - \pi/2]$$

and as $\underline{u}_2(x,\tau)$ satisfies the estimate (1.3.84) it is clearly the variation of $\underline{a}(\tau)$

which will determine the 'size' of $\underline{u}$.

We observe that $\underline{a}(\tau)$ becomes infinitely large as τ approaches τ_1 where

$$\tau_1 = \frac{\pi}{\underline{C}_2}.$$

Substituting for $\underline{C}_2$ by equation (1.3.94), therefore,

$$\tau_1 = \frac{\pi}{\underline{C}_2} = \frac{\pi}{\left\{(\tfrac{1}{2}\lambda^* I_3-\underline{k}_2)\ (I_1-\underline{k}_1)\right\}^{1/2}} \tag{1.3.95}$$

and τ_1 is an increasing function of both $\underline{k}_1$ and $\underline{k}_2$. This suggests that our 'best estimate' of t_b will follow choice of minimum values for both k_1 and k_2 .

However, if both k_1 and k_2 are 'small', we see from (1.3.95) that

$$\tau_1 \sim \frac{\pi}{(\tfrac{1}{2}\lambda^* I_1 I_3)^{1/2}} + D_1 k_1 + D_2 k_2 + \ldots . \tag{1.3.96}$$

and hence that it is the value of

$$\max(k_1, k_2)$$

which will determine a minimum value of τ_1 .

Next, at $\tau = \tau_0$, we have that

$$\begin{aligned} u(x,\tau_0) &= \omega^* + \epsilon^{1/2} u_1(x,\tau_0) + \epsilon u_2(x,\tau_0) \\ &= \omega^* + \epsilon^{1/2}\varphi^* a(\tau_0) + \epsilon u_2(x,\tau_0) . \end{aligned} \tag{1.3.97}$$

Our upper solution to u from Section 1.2.2, however, tells us that

$u \leq \omega^* - \psi + \psi_1$ from (1.2.49)

with the functions ψ and ψ_1 defined by (1.2.33) and (1.2.48) respectively.

As $\tau_0 = \epsilon^{1/2} t_0$, and we assume that inequality (1.2.49) holds at $t = t_0$, we have that

$$u(x, t_0) \leq \omega^* - \frac{C\varphi^* \epsilon^{1/2}}{\tau_0}$$

for some positive constant C.

Hence, to have a chance at satisfying our required initial condition (1.3.55), we must at least ensure that

$$\omega^* + \epsilon^{1/2}\varphi^* a(\tau_0) + \epsilon u_2(x,\tau_0) \leq \omega^* - \frac{C\varphi^*\epsilon^{1/2}}{\tau_0}$$

and consequently, as

$$a(\tau_0) = C_1 \tan\left[C_2\tau_0 - \frac{\pi}{2} \right],$$

that τ_0 is 'sufficiently small'.

Hence, if τ_0 is 'small', then

$$a(\tau_0) = C_1 \tan\left[C_2\tau_0 - \frac{\pi}{2}\right] \sim - C_1\left\{\frac{1}{C_2\tau_0} - \tfrac{1}{2}C_2\tau_0 + \ldots\right\} = -\frac{C_1}{C_2\tau_0} + \tfrac{1}{2}C_1C_2\tau_0 + \ldots$$

and from equations (1.3.92),

$$a(\tau_0) \sim \frac{-1}{(\tfrac{1}{2}\lambda^* I_3 - k_2)\tau_0} + \tfrac{1}{2}(I_1 - k_1)\tau_0 + \ldots . \qquad (1.3.98)$$

It follows, therefore, that if τ_0 is sufficiently small, then it will be k_2 which has the dominant influence on the size of u at $\tau = \tau_0$.

With these considerations in mind, we return to the problem of choosing k_1 and k_2

to allow us to satisfy the necessary inequality (1.3.91). With $d(\tau)$ as given by equation (1.3.75) and $\underline{c}_1$, $\underline{c}_2$ as defined in (1.3.92), inequality (1.3.91) requires that

$$\begin{aligned}
&\epsilon^{3/2}E_2\varphi^*\left[\, 1+|\underline{a}|^3\,\right] \\
&\quad + \lambda^* e^{\omega^*}\left\{ -\frac{\epsilon^{3/2}\underline{a}^3\varphi^{*3}}{6} + M_1M_2\epsilon^{3/2}E_1\varphi^{*2}\,|\underline{a}|\,[1+\underline{a}^2]\right\} \\
&\quad + \epsilon e^{\omega^*}\left\{ M_1\epsilon^{1/2}\,|\underline{a}|\,\varphi^* + M_2\epsilon E_1\varphi^*[1+\underline{a}^2] + \epsilon^{3/2}M_1M_2E_1\varphi^{*2}\,|\underline{a}|\,[1+\underline{a}^2]\right\} \\
&\qquad\qquad \le \epsilon\varphi^*\left[\, \underline{k}_1+\underline{k}_2\underline{a}^2\,\right]
\end{aligned}$$

$$\text{for } x\epsilon\Omega,\ \tau \ge \underline{\tau}_0 \tag{1.3.99}$$

and where we have assumed that $\underline{a}(\tau)$ satisfies the estimate (1.3.90).

To simplify investigation of this inequality, however, we note that there exists positive constants A_1, A_2 and A_3 for which the left hand side of inequality (1.3.99) is less than or equal to

$$\epsilon^{3/2}\varphi^*\{\, A_1\,|\underline{a}|^3 + A_2\,|\underline{a}| + A_3\,\}\ , \tag{1.3.100}$$

and satisfaction of inequality (1.3.99) is guaranteed if

$$\epsilon^{3/2}\varphi^*\{\, A_1\,|\underline{a}|^3 + A_2\,|\underline{a}| + A_3\,\} \le \epsilon\varphi^*[\, \underline{k}_1 + \underline{k}_2\underline{a}^2\,]$$

i.e. if

$$\epsilon^{1/2}\,\{\, A_1\,|\underline{a}|^3 + A_2\,|\underline{a}| + A_3\,\} \le \underline{k}_1 + \underline{k}_2\underline{a}^2 \tag{1.3.101}$$

for all considered τ .

We consequently define the constant A , by

$$A = A_1 + A_2 + A_3$$

and see that inequality (1.3.101) will be satisfied if

$$\epsilon^{1/2} A \mid \underline{a} \mid^3 \leq \underline{k}_1 + \underline{k}_2 \underline{a}^2 \qquad \text{for} \quad \mid \underline{a} \mid \geq 1$$

and (1.3.102)

$$\epsilon^{1/2} A \leq \underline{k}_1 + \underline{k}_2 \underline{a}^2 \qquad \text{for} \quad \mid \underline{a} \mid < 1 \ .$$

In line with (1.3.90) we allow $\mid \underline{a} \mid$ to take the value

$$\mid \underline{a} \mid = \epsilon^{-\gamma} \qquad \text{for some} \quad 0<\gamma<1/2 \tag{1.3.103}$$

in which case the first inequality of (1.3.102) requires that

$$\underline{k}_1 + \epsilon^{-2\gamma} \underline{k}_2 \geq \epsilon^{1/2 - 3\gamma} A \ . \tag{1.3.104}$$

To satisfy the second inequality of (1.3.102) for all $\mid \underline{a} \mid < 1$ we must also choose

$$\underline{k}_1 \geq \epsilon^{1/2} A \ . \tag{1.3.105}$$

Any $\underline{k}_1$ and $\underline{k}_2$ which satisfy (1.3.104) and (1.3.105) will consequently lead to satisfaction of inequality (1.3.99) and hence to the required inequality for $\underline{u}$ to be a lower solution to u , i.e. (1.3.54). However, to allow the 'best' upper estimate to t_b we look for a minimum value of $\max(\underline{k}_1, \underline{k}_2)$ and this is achieved by choosing

$$\underline{k}_1 \geq \epsilon^{1/4} A \quad \text{and} \quad \underline{k}_2 \geq \epsilon^{1/4-\gamma} A - \epsilon^{2\gamma}\underline{k}_1 - \epsilon^{1/4-\gamma} A' \quad \text{say} \tag{1.3.106}$$

for some positive constants A, A' and γ as in (1.3.103).

We now consider the second condition necessary for $\underline{u}$ to be a lower solution to u , inequality (1.3.55), which requires that

$$\underline{u}(x,\underline{t}_0) \leq u(x,\underline{t}_0) \qquad \text{for} \quad x\epsilon\Omega \ .$$

As in (1.3.97),

$$\underline{u}(x,\underline{\tau}_0) = \omega^*(x) + \epsilon^{1/4}\varphi^*(x)\underline{a}(\underline{\tau}_0) + \epsilon\underline{u}_2(x,\underline{\tau}_0)$$

and we may estimate $\underline{u}_2(x,\underline{\tau}_0)$ by (1.3.84) to see that

$$\underline{u}(x,\underline{\tau}_0) \leq \omega^*(x) + \epsilon^{1/4}\varphi^*(x)\underline{a}(\underline{\tau}_0) + \epsilon E_1\varphi^*(x)\left[1+\underline{a}^2(\underline{\tau}_0)\right] \ \text{for} \quad x\epsilon\Omega \ . \tag{1.3.107}$$

Having established that $\underline{\tau}_0$ must be small, we see from (1.3.98) that in this case there exists a positive constant M such that

$$-\frac{1}{(\frac{1}{2}\lambda^* I_3-\underline{k}_2)\underline{\tau}_0} \leq \underline{a}(\underline{\tau}_0) \leq -\frac{1}{(\frac{1}{2}\lambda^* I_3-\underline{k}_2)\underline{\tau}_0} + M\underline{\tau}_0, \tag{1.3.108}$$

and

$$\underline{a}^2(\underline{\tau}_0) \leq \frac{1}{(\frac{1}{2}\lambda^* I_3-\underline{k}_2)^2\underline{\tau}_0^2}.$$

We may estimate from (1.3.107), that

$$\underline{u}(x,\mathcal{I}_0) \le \omega^* + \epsilon^{1/2}\varphi^*\left\{\frac{-1}{(1/2\lambda^* I_3-\underline{k}_2)\mathcal{I}_0} + M\mathcal{I}_0\right\}$$
$$+ \epsilon E_1\varphi^*\left\{1 + \frac{1}{(1/2\lambda^* I_3-\underline{k}_2)^2\mathcal{I}_0^2}\right\}$$

so that

$$\underline{u}(x,\mathcal{I}_0) \le \omega^* - \frac{\epsilon^{1/2}\varphi^*}{(1/2\lambda^* I_3-\underline{k}_2)\mathcal{I}_0} + \epsilon^{1/2}\varphi^* M\mathcal{I}_0$$
$$+ \epsilon E_1\varphi^* + \frac{\epsilon E_1\varphi^*}{(1/2\lambda^* I_3-\underline{k}_2)^2\mathcal{I}_0^2}$$
$$\text{for} \quad x\in\Omega. \tag{1.3.109}$$

We must also ensure that $\underline{a}(\mathcal{I}_0)$ satisfies that estimate (1.3.103) for some $0<\gamma<1/2$,

and hence require that

$$|\underline{a}(\mathcal{I}_0)| \le \frac{1}{[1/2\lambda^* I_3-\underline{k}_2]\mathcal{I}_0} \le \epsilon^{-\gamma}. \tag{1.3.110}$$

This will be satisfied as required if $\mathcal{I}_0 - C\epsilon^{\gamma}$ for some suitably large constant C .

Finally, it is clear that

$$\frac{1}{1/2\lambda^* I_3-\underline{k}_2} > \frac{1}{1/2\lambda^* I_3} + \frac{\underline{k}_2}{(1/2\lambda^* I_3)^2} \tag{1.3.111}$$

and, if $\underline{k}_2$ is 'small', that

$$\frac{1}{(\tfrac{1}{2}\lambda^* I_3 - \underline{k}_2)^2} \le \frac{R}{(\tfrac{1}{2}\lambda^* I_3)^2} \qquad \text{say,} \tag{1.3.112}$$

for some positive constant R .

Using (1.3.111) and (1.3.112) in (1.3.109) yields that

$$\begin{aligned} \underline{u}(x, \tau_0) \le \omega^* &- \frac{\epsilon^{1/2}\varphi^*}{(\tfrac{1}{2}\lambda^* I_3)\underline{\tau}_0} - \frac{\epsilon^{1/2}\varphi^* \underline{k}_2}{(\tfrac{1}{2}\lambda^* I_3)^2 \underline{\tau}_0} \\ &+ \epsilon^{1/2}\varphi^* M \underline{\tau}_0 + \epsilon E_1 \varphi^* + \frac{\epsilon E_1 R \varphi^*}{(\tfrac{1}{2}\lambda^* I_3)^2 \underline{\tau}_0^2} \end{aligned}$$

$$\text{for} \quad x \in \Omega . \tag{1.3.113}$$

In order to verify (1.3.55), however, we must also estimate $u(x, \underline{\tau}_0)$. To do this, we make use of the lower estimate to u derived in Section 1.3.1. Recalling the concluding inequality of that section, inequality (1.3.52), we have that

$$\begin{aligned} u(x,t) \ge \omega^* &- \frac{\varphi^*}{(\tfrac{1}{2}\lambda^* I_3 t)} - \frac{\varphi^* B_3 \bar{t}}{(\tfrac{1}{2}\lambda^* I_3)\, t^2} \\ &- \frac{B_2 \delta \varphi^*}{t} - \frac{B_2 B_3 \delta \varphi^* \bar{t}}{t^2} \\ &- V_1 \varphi^* \left[\frac{1}{t^2} + \frac{2 B_3 \bar{t}}{t^3} + \frac{B_3^2 \bar{t}^2}{t^4} \right] \end{aligned}$$

$$\text{for} \quad x \in \Omega \tag{1.3.52}$$

provided $t \ge \bar{t} \ge \alpha_0'/\delta$ and $\bar{t}/t$ is 'small'.

Further, $\delta > 0$ is chosen small and $\alpha_0' = \dfrac{2\alpha_0 V_1}{\alpha_1 [1 + \delta (k\lambda^* I_3)]}$ where α_0 is suitably large.

Hence, if $\underline{t}_0 = \epsilon^{-1/2}\underline{\tau}_0 \geq \bar{t} \geq \alpha_0'/\delta$ and $\bar{t}/\underline{t}_0$ is small

where $\underline{\tau}_0 = C\epsilon^{\gamma}$, then inequality (1.3.52) may be used to estimate the size of

$u(x,\underline{t}_0)$, and

$$
\begin{aligned}
u(x,\underline{t}_0) \geq \omega^* &- \frac{\varphi^*}{(\tfrac{1}{2}\lambda^* I_3 \underline{t}_0)} - \frac{\varphi^* B_3 \bar{t}}{(\tfrac{1}{2}\lambda^* I_3)\underline{t}_0^{\,2}} \\
&- \frac{B_2\delta\varphi^*}{\underline{t}_0} - \frac{B_2 B_3 \delta\varphi^* \bar{t}}{\underline{t}_0^{\,2}} \\
&- V_1\varphi^* \left[\frac{1}{\underline{t}_0^{\,2}} + \frac{2B_3\bar{t}}{\underline{t}_0^{\,3}} + \frac{B_3^2\bar{t}^2}{\underline{t}_0^{\,4}} \right]
\end{aligned}
$$

for $x \in \Omega$. (1.3.114)

Combining the results of inequalities (1.3.113) and (1.3.114) we see that inequality (1.3.55) will be satisfied at $\underline{t}_0 = \epsilon^{-1/2}\underline{\tau}_0$ if

$$
\begin{aligned}
\underline{u}(x,\underline{\tau}_0) &\leq \omega^* - \frac{\epsilon^{1/2}\varphi^*}{(\tfrac{1}{2}\lambda^* I_3)\underline{\tau}_0} - \frac{\epsilon^{1/2}\varphi^* k_2}{(\tfrac{1}{2}\lambda^* I_3)^2 \underline{\tau}_0} \\
&\quad + \epsilon^{1/2}\varphi^* M \underline{\tau}_0 + \epsilon E_1 \varphi^* + \frac{\epsilon E_1 R \varphi^*}{(\tfrac{1}{2}\lambda^* I_3)^2 \underline{\tau}_0^{\,2}} \\
&\leq \omega^* - \frac{\varphi^*}{(\tfrac{1}{2}\lambda^* I_3)\underline{t}_0} - \frac{\varphi^* B_3 \bar{t}}{(\tfrac{1}{2}\lambda^* I_3)\underline{t}_0^{\,2}} - \frac{B_2\delta\varphi^*}{\underline{t}_0} \\
&\quad - \frac{B_2 B_3 \delta \varphi^* \bar{t}}{\underline{t}_0^{2}} - V_1\varphi^* \left\{ \frac{1}{\underline{t}_0^{\,2}} + \frac{2B_3\bar{t}}{\underline{t}_0^{\,3}} + \frac{B_3^2\bar{t}^2}{\underline{t}_0^{\,4}} \right\} \\
&\leq u(x,\underline{t}_0)
\end{aligned}
$$

for $x \in \Omega$ (1.3.115)

where $\underline{t}_0 \geq \bar{t} \geq \alpha_0'/\delta$, $\bar{t}/\underline{t}_0$ is small, with $\underline{t}_0 = \epsilon^{-1/2}\underline{\tau}_0$ and $\underline{\tau}_0 = C\epsilon^{\gamma}$ for some

$0<\gamma<\tfrac{1}{2}$.

On substituting for $t_0 = \epsilon^{-1/2}\mathrm{I}_0$, inequality (1.3.115) is seen to be equivalent to requiring that

$$\frac{\epsilon^{1/2}\varphi^* k_2}{(\tfrac{1}{2}\lambda^* I_3)^2\mathrm{I}_0} \geq \epsilon^{1/2}\varphi^* M\,\mathrm{I}_0 + \epsilon E_1\varphi^* + \frac{\epsilon E_1 R\varphi^*}{(\tfrac{1}{2}\lambda^* I_3)^2\mathrm{I}_0^{\,2}}$$
$$+ \frac{\epsilon\varphi^* B_3\bar{t}}{(\tfrac{1}{2}\lambda^* I_3)\,\mathrm{I}_0^{\,2}} + \frac{\epsilon^{1/2}B_2\delta\varphi^*}{\mathrm{I}_0} + \frac{B_2B_3\epsilon\delta\bar{t}\varphi^*}{\mathrm{I}_0^{\,2}}$$
$$+ V_1\varphi^*\left\{\frac{\epsilon}{\mathrm{I}_0^{\,2}} + \frac{2B_3\epsilon^{3/2}\bar{t}}{\mathrm{I}_0^{\,3}} + \frac{B_3^2\epsilon^2\bar{t}^2}{\mathrm{I}_0^{\,4}}\right\}$$

which suggests that we must choose k_2 to satisfy

$$k_2 \geq (\tfrac{1}{2}\lambda^* I_3)^2\left\{M\,\mathrm{I}_0^{\,2} + \epsilon^{1/2}E_1\mathrm{I}_0 + \frac{\epsilon^{1/2}E_1 R}{(\tfrac{1}{2}\lambda^* I_3)^2\mathrm{I}_0}\right.$$
$$+ \frac{\epsilon^{1/2}B_3\bar{t}}{(\tfrac{1}{2}\lambda^* I_3)\,\mathrm{I}_0} + B_2\delta + \frac{B_2B_3\epsilon^{1/2}\delta\bar{t}}{\mathrm{I}_0}$$
$$\left. + V_1\left[\frac{\epsilon^{1/2}}{\mathrm{I}_0} + \frac{2B_3\epsilon\bar{t}}{\mathrm{I}_0^{\,2}} + \frac{B_3^2\epsilon^{3/2}\bar{t}^2}{\mathrm{I}_0^{\,3}}\right]\right\}. \tag{1.3.116}$$

We have taken $\bar{t}/t_0$ to be small and hence assume that

$$\bar{t}/t_0 = \epsilon^p \quad \text{for some } p>0 \ . \tag{1.3.117}$$

It follows, therefore, that $\bar{t} = t_0\epsilon^p = \epsilon^{-1/2+p}\mathrm{I}_0$. We also require that $\bar{t} \geq \alpha_0'/\delta$ and this is satisfied if we take

$$\delta = \alpha_0'/\bar{t} = \frac{\alpha_0'\epsilon^{1/2-p}}{\mathrm{I}_0} \tag{1.3.118}$$

for any large α_0' .

With (1.3.117) and (1.3.118), inequality (1.3.116) requires that k_2 be chosen such that

$$k_2 \geq (\tfrac{1}{2}\lambda^* I_3)^2 \left\{ M\, \mathcal{I}_0^{\,2} + \epsilon^{1/2} E_1 \mathcal{I}_0 + \frac{\epsilon^{1/2} E_1 R}{(\tfrac{1}{2}\lambda^* I_3)^2 \mathcal{I}_0} \right.$$
$$+ \frac{B_3 \epsilon^p}{(\tfrac{1}{2}\lambda^* I_3)} + \frac{B_2 \alpha_0' \epsilon^{1/2 - p}}{\mathcal{I}_0} + \frac{B_2 B_3 \alpha_0' \epsilon^{1/2}}{\mathcal{I}_0}$$
$$\left. + V_1 \left[\frac{\epsilon^{1/2}}{\mathcal{I}_0} + \frac{2 B_3 \epsilon^{1/2 + p}}{\mathcal{I}_0} + \frac{B_3^2 \epsilon^{1/2 + 2p}}{\mathcal{I}_0} \right] \right\}$$

which is clearly satisfied if k_2 is chosen such that

$$k_2 \geq D_1 \mathcal{I}_0^{\,2} + \frac{D_2 \epsilon^{1/2}}{\mathcal{I}_0} + D_3 \epsilon^p + \frac{D_4 \alpha_0' \epsilon^{1/2 - p}}{\mathcal{I}_0} \tag{1.3.119}$$

for positive constants D_1, D_2, D_3 and D_4 , and any suitably large α_0' .

To summarise our requirements of k_1 and k_2 , we see from (1.3.103) and (1.3.106), that if $0<\gamma<\tfrac{1}{2}$ with $\max |\underline{a}(t)| \sim \epsilon^{-\gamma}$ then

$$k_1 \geq \epsilon^{1/2} A \quad \text{and} \quad k_2 \geq \epsilon^{1/2 - \gamma} A \tag{1.3.120}$$

and from (1.3.119), as $\mathcal{I}_0 \sim C\epsilon^{\gamma}$, that

$$k_2 \geq D.\max\left[\epsilon^{2\gamma}, \epsilon^{1/2-\gamma}, \epsilon^p, \alpha_0' \epsilon^{1/2 - (p+\gamma)}\right]$$

for some $p>0$ and any large α_0'.

It follows that for any $p>0$, then $\epsilon^{1/2-(\gamma+p)}>\epsilon^{1/2-\gamma}$ and our requirement of k_2 becomes

$$k_2 \geq D.\max\left[\epsilon^{2\gamma}, \epsilon^{p}, \alpha_0'\epsilon^{1/2-(\gamma+p)}\right] \tag{1.3.121}$$

for $0<\gamma<1/2$ such that

$$|a(\tau)| \leq \epsilon^{-\gamma} \text{ for } \tau \text{ in the considered range,} \tag{1.3.122}$$

with $p>0$ and α_0' sufficiently large (but unrelated to the size of ϵ).

Hence we conclude that the function $\underline{u}(x,t)$ defined by (1.3.58) as

$$\begin{aligned}\underline{u}(x,t) &= \omega^*(x) + \epsilon^{1/2}\underline{u}_1(x,t) + \epsilon\underline{u}_2(x,t)\\ &= \omega^*(x) + \epsilon^{1/2}\underline{a}(\tau)\varphi^*(x) + \epsilon\underline{u}_2(x,\tau)\end{aligned}$$

is, with $\underline{a}(\tau)$ and $\underline{u}(x,\tau)$ as described and k_1, k_2 satisfying (1.3.120) and (1.3.121) respectively, a lower solution to $u(x,t)$ for all $t\geq t_0 = \epsilon^{-1/2}\tau_0$ with $\tau_0 = C\epsilon^{\gamma}$ and $t\leq t_1$ where t_1 is the first time after t_0 which inequality (1.3.122) fails.

1.3.3 Time region III.

Section 1.3.2 has established a lower solution to u which exists until the first time after $t=\underline{t}_0$ (say $t=\underline{t}_1$) at which inequality (1.3.122) fails. This alone, however, does not yet allow an estimate for t_b because, while u must be greater than $\underline{u}$ at $t=\underline{t}_1$, we do not yet have any evidence as to whether u blows up before or after this time. In this section, we establish the evidence necessary to allow us to bound t_b closely from above. We begin by estimating the size of $u(x,\underline{t}_1)$.

Inequality (1.3.122) requires that

$$|\underline{a}(\tau)| \le \epsilon^{-\gamma}$$

i.e. that

$$|\underline{C}_1 \tan(\underline{C}_2\tau - \pi/2)| \le \epsilon^{-\gamma}$$

on substituting for $\underline{a}(\tau)$ by equation (1.3.166).

The time $\underline{t}_1 = \epsilon^{-1/4}\underline{\tau}_1$ is taken to the first time after $t=\underline{t}_0$ at which this inequality fails, and hence $\underline{\tau}_1 = \frac{\pi}{\underline{C}_2} - \underline{\tau}_F$ for some 'small' $\underline{\tau}_F$.

Further,

$$\underline{a}(\underline{\tau}_1) = \underline{C}_1 \tan[\pi/2 - \underline{C}_2\underline{\tau}_F] \le \frac{\underline{C}_1}{\underline{C}_2\underline{\tau}_F}$$

if $\underline{\tau}_F$ is sufficiently small.

Hence, we see that a maximal value of $\underline{\tau}_1$ is

$$\underline{\tau}_1 = \frac{\pi}{\underline{C}_2} - C\epsilon^{\gamma} \tag{1.3.123}$$

where $C \geq \underline{C}_1/\underline{C}_2$.

At $t=\underline{t}_1=\epsilon^{-1/2}\underline{\tau}_1$, $\underline{u}(x,t)$ as defined by (1.3.158) is a lower solution to u and

$$u(x,\underline{t}_1) \geq \underline{u}(x,\underline{\tau}_1) = \omega^* + \epsilon^{1/2}\varphi^*\underline{a}(\underline{\tau}_1) + \epsilon\underline{u}_2(x,\underline{\tau}_1) \ .$$

We may also use (1.3.84) to estimate $\underline{u}_2(x,\tau_1)$ so that

$$u(x,\underline{t}_1) \geq \omega^* + \epsilon^{1/2}\varphi^*\underline{a}(\underline{\tau}_1) - \epsilon E_1\varphi^*[1 + \underline{a}^2(\underline{\tau}_1)] \qquad \text{for} \quad x\in\Omega \ .$$

Taking $\underline{a}(\underline{t}_1) = \epsilon^{-\gamma}$, we see that

$$u(x,\underline{t}_1) \geq \omega^* + \varphi^*\epsilon^{1/2-\gamma} - \epsilon E_1\varphi^*[1 + \epsilon^{-2\gamma}] \qquad \text{for} \quad x\in\Omega \tag{1.3.124}$$

where $\underline{t}_1 = \epsilon^{-1/2}\underline{\tau}_1$ and τ_1 is defined in (1.3.123).

We next consider times $t\geq\underline{t}_1$.

In Section 1.2.1 it was established that, if the function $v(x,t)$ is defined (as in

(1.2.56)) as

$$v(x,t) = u(x,t) - \omega^*(x) \qquad \text{in} \quad \Omega,\ t>0,$$

and $a^*(t)$ is defined (as in (1.2.8)) by

$$a^*(t) = \int_\Omega \varphi^*(x)\, v(x,t)\, dx$$

where φ^* is the solution to (1.2.3)-(1.2.5), then

$$a_t^* \geq k_1 + k_2 a^{*2} \qquad \text{for} \quad t>0,$$

where $k_1 = (\lambda-\lambda^*)\,.\min\left[1,\ \exp\left\{\inf_{x\in\Omega} u_0(x)\right\}\right]$

and $$k_2 = \tfrac{1}{2}k\lambda^* \begin{cases} 1 & \text{if } \inf_{x\in\Omega} u_0(x) \geq 0 \\ \exp\left\{\inf_{x\in\Omega} u_0(x)\right\} & \text{if } \inf_{x\in\Omega} u_0(x) < 0 \end{cases}$$

(c.f. (1.2.17)-(1.2.19)).

Hence as $k_1>0$ and we take $k_2=k$, it follows that

$$a_t^* \geq k a^{*2} \qquad \text{for} \quad t > 0\ . \tag{1.3.125}$$

Now if $u(x,t)$ exists beyond $t = \underline{t}_1$, i.e. if

$$t_b > \underline{t}_1\ ,$$

we may integrate (1.3.125) from $\underline{t}_1$ to t for any $\underline{t}_1<t<t_b$ and find that,

as $a^*(\underline{t}_1) \geq 0$, then

$$t \leq \underline{t}_1 + \frac{1}{ka^*(\underline{t}_1)} \tag{1.3.126}$$

for any $\underline{t}_1<t<t_b$.

Now,

$$\begin{aligned} a^*(\underline{t}_1) &= \int_\Omega \varphi^* v(x,\underline{t}_1)\,dx \\ &= \int_\Omega \varphi^* \big(u(x,\underline{t}_1) - \omega^*(x)\big)dx \end{aligned} \tag{1.3.127}$$

and as $\underline{u}(x,t)$ is a lower solution to u at $t=\underline{t}_1$, we see from inequality (1.3.124) that

$$u(x,\underline{t}_1) - \omega^*(x) \geq \varphi^*\epsilon^{\frac{1}{2}-\gamma} - \epsilon E_1\varphi^*[1+\epsilon^{-2\gamma}] \ .$$

It follows, as $\frac{1}{2}>\gamma>0$, that there exists a positive constant E such that

$$u(x,\underline{t}_1) - \omega^*(x) \geq E\varphi^*(x)\epsilon^{\frac{1}{2}-\gamma} \quad \text{throughout } \Omega \ . \tag{1.3.128}$$

Substituting (1.3.128) into (1.3.127) then yields that

$$\begin{aligned} a^*(\underline{t}_1) &= \int_\Omega \varphi^*[u(x,\underline{t}_1) - \omega^*(x)]\,dx \\ &\geq \int_\Omega E\varphi^{*2}\epsilon^{\frac{1}{2}-\gamma}dx \end{aligned}$$

and

$$a^*(\underline{t}_1) \geq E' e^{1/2-\gamma} \tag{1.3.129}$$

for some positive constant E' .

Substituting for $a^*(\underline{t}_1)$ by (1.3.129) in (1.3.126) then establishes that

$$t \leq \underline{t}_1 + \frac{\epsilon^{-1/2+\gamma}}{kE'} \qquad \text{for all} \quad \underline{t}_1 < t < t_b$$

and hence, on letting $t \to t_b$, that

$$t_b \leq \underline{t}_1 + \frac{\epsilon^{-1/2+\gamma}}{kE'} \ . \tag{1.3.130}$$

Inequality (1.3.130) is clearly also valid if $t_b \leq \underline{t}_1$ and we conclude that

$$t_b \leq \underline{t}_1 + \frac{\epsilon^{-1/2+\gamma}}{kE'}$$

where $\underline{t}_1 = \epsilon^{-1/2}\underline{\tau}_1$ and $\underline{\tau}_1$ is given by (1.3.123).

Hence,

$$\underline{\tau}_1 = \frac{\pi}{\underline{C}_2} - C\epsilon^{\gamma}$$

for positive constant C and $\underline{C}_2$ defined by equation (1.3.94), i.e.

$$\mathcal{L}_2 = \{[\tfrac{1}{2}\lambda^* I_3 - k_2][I_1 - k_1]\}^{\frac{1}{2}} .$$

As both k_1 and k_2 are small, it follows that

$$\mathcal{I}_1 \le \frac{\pi}{(\tfrac{1}{2}\lambda^* I_1 I_3)^{\frac{1}{2}}} + D_1 k_1 + D_2 k_2 - C\epsilon^{\gamma}$$

for positive constants D_1, D_2 as in (1.3.96), and k_1, k_2 satisfying (1.3.120), (1.3.121) respectively.

Hence,

$$\begin{aligned} t_b &\le t_1 + \frac{\epsilon^{-\frac{1}{2}+\gamma}}{E'k} \\ &= \epsilon^{-\frac{1}{2}}\mathcal{I}_1 + \frac{\epsilon^{-\frac{1}{2}+\gamma}}{E'k} \\ &\le \frac{\pi\epsilon^{-\frac{1}{2}}}{(\tfrac{1}{2}\lambda^* I_1 I_3)^{\frac{1}{2}}} + D_1 k_1 \epsilon^{-\frac{1}{2}} + D_2 k_2 \epsilon^{-\frac{1}{2}} - C\epsilon^{-\frac{1}{2}+\gamma} + \frac{\epsilon^{-\frac{1}{2}+\gamma}}{E'k} . \end{aligned} \tag{1.3.131}$$

Finally, on substituting minimal values of k_1, k_2 from (1.3.120), (1.3.121), we conclude that

$$\begin{aligned} t_b \le \frac{\pi\epsilon^{-\frac{1}{2}}}{(\tfrac{1}{2}\lambda^* I_1 I_3)^{\frac{1}{2}}} &+ AD_1 + \left(\frac{1}{E'k} - C\right)\epsilon^{-\frac{1}{2}+\gamma} \\ &+ D_2 . D . \max\left[\epsilon^{-\frac{1}{2}+2\gamma}, \epsilon^{-\frac{1}{2}+p}, \alpha_0'\epsilon^{-(\gamma+p)}\right] \end{aligned} \tag{1.3.132}$$

for any $0<\gamma<\tfrac{1}{2},\ p>0$ and $\alpha_0'\epsilon^{\frac{1}{2}-(\gamma+p)}$ 'small' (as we have taken k_2 to be small).

Hence, as α_0' may be chosen as large as required for suitably small ϵ if we set

$$\alpha_0' \sim e^{-q} \text{ for any } q > 0 \,, \tag{1.3.133}$$

(1.3.132) reduces to

$$t_b \le \frac{\pi e^{-1/2}}{[1/2\lambda^* I_1 I_3]^{1/2}} + \left[\frac{1}{E'k} - C\right] e^{-1/2+\gamma} + D_2 . D . \max\left[e^{-1/2+2\gamma},\ e^{-1/2+p},\ e^{-(\gamma+p+q)}\right] \tag{1.3.134}$$

where we take $0<\gamma<1/2,\ p, q > 0$ and $\gamma+p+q<1/2$.

We see, therefore, that there exists $s_1>0$ such that

$$t_b \le \frac{\pi(\lambda-\lambda^*)^{-1/2}}{[1/2\lambda^* I_1 I_3]^{1/2}} [1+e^{s_1}] \qquad \text{for } \lambda\to\lambda^*+ \,. \tag{1.3.135}$$

Section 1.4 A lower bound on t_b

1.4.1 An upper solution in time region I

As indicated in Section 1.1, the purpose of Section 1.4 is to identify an upper solution to u from which we may derive a lower bound for t_b , the blow up time of u .

In order to establish this upper solution, we must assume, as in section 1.2.2, that the conditions (1.2.21) and (1.2.22) hold, i.e. that

$$u_0(x) < \omega^*(x) \quad \text{in} \quad \Omega \tag{1.2.21}$$

and either

$$\frac{\partial u_0}{\partial n} > \frac{\partial \omega^*}{\partial n}, \quad \text{or} \quad u_0 < \omega^* \quad \text{on} \quad \partial\Omega. \tag{1.2.22}$$

It was shown in Section 1.2.2 that if conditions (1.2.21), (1.2.22) are satisfied, then there exits an upper solution to u which remains finite (and less than ω^*) at least until some time $t \sim t_1(\lambda-\lambda^*)^{-1/2}$.

We proceed in this section by finding a similar upper solution which is much "closer" to u , i.e. we seek to establish a minimum upper solution of the particular form considered.

Following the approach of Sections 1.2.2 and 1.3.1 we consider $u^*(x,t)$ the solution to (1.2.23-(1.2.25) and find that $\hat{u}(x,t)$ defined as

$$\hat{u} = \omega^* - u^*$$

satisfies the problem (1.2.27) - (1.2.29), i.e.

$$\hat{u} = -\lambda^* exp(\omega^* - \hat{u}) + \nabla^2\hat{u} - \nabla^2\omega^* \quad \text{in} \quad \Omega \ t>0, \tag{1.2.27}$$

$$\hat{u}(x,0) = \omega^*(x) - u_0(x) \quad \text{in} \quad \Omega, \tag{1.2.28}$$

and $$\frac{\partial\hat{u}}{\partial n} + \beta\hat{u} = 0 \quad \text{on} \quad \partial\Omega, \ t>0. \tag{1.2.29}$$

Conditions (1.2.21), (1.2.22) again ensure that the condition (1.2.30) remains valid, so that

$$\hat{u}(x,t) \geq 0 \quad \text{throughout} \quad \Omega, \ t>0$$

which allows us to estimate

$$\lambda^* e^{\omega^*}[1 - e^{-\hat{u}}] \geq \lambda^* e^{\omega^*}[\hat{u} - \tfrac{1}{2}\hat{u}^2].$$

Applying this estimate in (1.2.27) then yields that

$$\hat{u}_t \geq \nabla^2\hat{u} + \lambda^* e^{\omega^*}[\hat{u} - \tfrac{1}{2}\hat{u}^2] \quad \text{in} \quad \Omega, \ t>0.$$

Hence, if it can be established that there exists some function $\Phi_0(x,t)$ satisfying

$$\frac{\partial\Phi_0}{\partial t} \leq \nabla^2\Phi_0 + \lambda^* e^{\omega^*}[\Phi_0 - \tfrac{1}{2}\Phi_0^2] \quad \text{in} \quad \Omega, \ t>0, \tag{1.4.1}$$

$$\Phi_0(x,0) \leq \hat{u}(x,0) = \omega^*(x) - u_0(x) \quad \text{in} \quad \Omega, \tag{1.4.2}$$

$$\frac{\partial\Phi_0}{\partial n} + \beta\Phi_0 \leq 0 \quad \text{on} \quad \partial\Omega, \ t>0, \tag{1.4.3}$$

then Φ_0 will be a lower solution to $\hat{u} = \omega^* - u^*$, and $\omega^* - \Phi_0$ will be an upper solution to u^*, for $t>0$.

Motivated by the analysis of Section 1.2.2 and by the asymptotic estimate of u in this region, we initially choose Φ_0 as

$$\Phi_0(x,t) = \frac{\mu_0\varphi^*}{(t+t_0)} - \frac{z_0(x)}{(t+t_0)^2} \tag{1.4.4}$$

for some constants μ_0, $t_0>0$ and function $z_0(x)$ yet to be determined.

Again, φ^* is taken as the solution to (1.2.3), (1.2.4) satisfying (1.2.34). With this Φ_0, inequality (1.4.1) requires that

$$-\left\{\frac{\mu_0\varphi^*}{(t+t_0)^2} - \frac{2z_0}{(t+t_0)^3}\right\} \le -\frac{1}{(t+t_0)^2}\left\{\nabla^2 z_0 + \lambda^* e^{\omega^*} z_0 + \tfrac{1}{2}\lambda^* e^{\omega^*}\left[\mu_0\varphi^* - \frac{z_0}{(t+t_0)}\right]^2\right\}$$

for $x\in\Omega,\ t>0$

which is satisfied if

$$\nabla^2 z_0 + \lambda^* e^{\omega^*} z_0 \le \mu_0\varphi^*\{1 - \tfrac{1}{2}\mu_0\lambda^* e^{\omega^*}\varphi^*\} - \frac{2z_0}{(t+t_0)}\{1 - \tfrac{1}{2}\mu_0\lambda^* e^{\omega^*}\varphi^*\} - \frac{\tfrac{1}{2}\lambda^* e^{\omega^*} z_0^2}{(t+t_0)^2}. \tag{1.4.5}$$

We consequently choose z_0 such that

$$\nabla^2 z_0 + \lambda^* e^{\omega^*} z_0 = \mu_0 \varphi^* \{1 - \tfrac{1}{2}\mu_0 \lambda^* e^{\omega^*} \varphi^*\} - e_0(x) \quad \text{in} \quad \Omega \tag{1.4.6}$$

with

$$\frac{\partial z_0}{\partial n} + \beta z_0 = 0 \quad \text{on} \quad \partial\Omega \tag{1.4.7}$$

for some $e_0(x) \geq 0$ yet to be determined and with

$$z_0(x) \geq 0 \quad \text{throughout} \quad \Omega. \tag{1.4.8}$$

From the Friedholm alternative such z_0 requires that

$$\int_\Omega \varphi^* \{\mu_0 \varphi^* [1 - \tfrac{1}{2}\mu_0 \lambda^* e^{\omega^*} \varphi^*] - e_0\} dx = 0$$

and hence that

$$\int_\Omega \varphi^* e_0 = \mu_0 [1 - \tfrac{1}{2}\mu_0 \lambda^* I_3] \tag{1.4.9}$$

with I_3 as defined in (1.2.58), i.e.

$$I_3 = \int_\Omega \varphi^{*3} e^{\omega^*}.$$

Equation (1.4.9) is satisfied if we choose

$$e_0(x) = \mu_0 \varphi^* [1 - \tfrac{1}{2}\mu_0 \lambda^* I_3] \tag{1.4.10}$$

(in light of (1.2.34))

in which case inequality (1.4.5) reduces to

$$e_0(x) \geq \frac{2z_0}{(t+t_0)}\{1 - ½\mu_0\lambda^* e^{\omega^*}\varphi^*\} + ½\lambda^* e^{\omega^*}\frac{z_0^2}{(t+t_0)^2} \quad \text{for} \quad x\epsilon\Omega,\ t>0. \qquad (1.4.11)$$

In addition to (1.4.11), $\Phi_0(x,t)$ will only be a lower solution to $\hat{u}$ if conditions (1.4.2) and (1.4.3) are also satisfied, i.e. if

$$\Phi_0(x,0) - \frac{\mu_0\varphi^*(x)}{t_0} - \frac{z_0(x)}{t_0^2} \leq \omega^*(x) - u_0(x) \quad \text{for} \quad x \text{ in } \Omega \qquad (1.4.12)$$

and

$$\frac{\partial\Phi_0}{\partial n} + \beta\Phi_0 \leq 0 \quad \text{for} \quad x \text{ on } \partial\Omega. \qquad (1.4.3)$$

Condition (1.4.3) is automatic in light of (1.2.4) and (1.4.7) and it remains to choose μ_0 and t_0 to satisfy inequalities (1.4.11) and (1.4.12).

Inequality (1.4.11) requires that

$$e_0(x) - \mu_0\varphi^*[1 - \mu_0½\lambda^* I_3] > 0 \quad \text{for} \quad x\epsilon\Omega$$

and we must therefore choose μ_0 to satisfy

$$\mu_0 < [½\lambda^* I_3]^{-1}. \qquad (1.4.13)$$

Substituting for $e_0(x)$ by equation (1.4.10) in the requirement (1.4.11) also indicates that μ_0 and t_0 must be chosen such that

$$\mu_0\varphi^*[1 - \mu_0 \tfrac{1}{2}\lambda^* I_3] \geq \frac{2z_0}{(t+t_0)}\{1 - \tfrac{1}{2}\mu_0\lambda^* e^{\omega^*}\varphi^*\} + \frac{\tfrac{1}{2}\lambda^* e^{\omega^*} z_0^2}{(t+t_0)^2}$$

$$\text{for } x \text{ in } \Omega,\ t>0. \tag{1.4.14}$$

The left hand side of this inequality is a maximum with respect to constant μ_0, if

$$\mu_0 = \tfrac{1}{2}[\tfrac{1}{2}\lambda^* I_3]^{-1} \tag{1.4.15}$$

and this choice satisfies the condition (1.4.13) as required.

Further, with this μ_0, equation (1.4.10) becomes

$$e_0(x) = \tfrac{1}{4}[\tfrac{1}{2}\lambda^* I_3]^{-1}\varphi^*(x)\,. \tag{1.4.16}$$

On substituting for this $e_0(x)$ in equation (1.4.6) we see that

$$\begin{aligned}\nabla^2 z_0 + \lambda^* e^{\omega^*} z_0 &= \mu_0\varphi^*[\tfrac{1}{2}\lambda^*\mu_0(I_3 - e^{\omega^*}\varphi^*)] \\ &= \tfrac{1}{2}\lambda^*\mu_0^2\varphi^*[I_3 - e^{\omega^*}\varphi^*]\end{aligned} \quad \text{for x in } \Omega \tag{1.4.17}$$

with $$\frac{\partial z_0}{\partial n} + \beta z_0 = 0 \quad \text{on } \partial\Omega \tag{1.4.18}$$

which, in light of (1.4.15), suggests that z_0 may be chosen such that

$$z_0(x) \leq Z_0\varphi^*(x) \quad \text{for } x \text{ in } \Omega \tag{1.4.19}$$

and some positive constant Z_0.

Inequality (1.4.14) requires that

$$\mu_0\varphi^*[1 - \mu_0½\lambda^* I_3] \geq \frac{2z_0}{(t+t_0)}\{1 - \mu_0½\lambda^* e^{\omega^*}\varphi^*\} + \frac{½\lambda^* e^{\omega^*} z_0^2}{(t+t_0)^2}$$

for $x\epsilon\Omega$

and this will be satisfied, for positive μ_0, z_0 and all $t>0,$ if

$$\mu_0\varphi^*[1 - \mu_0½\lambda^* I_3] \geq \frac{2z_0}{t_0} + \frac{½\lambda^* e^{\omega^*} z_0^2}{t_0^2} \text{ for } x \text{ in } \Omega \, . \tag{1.4.20}$$

Using (1.4.19) to estimate $z_0(x)$ in (1.4.20), and substituting for μ_0 by (1.4.15), leads to the conclusion that inequality (1.4.14) will be satisfied as required for all x in $\Omega, t>0,$ if

$$¼(½\lambda^* I_3)^{-1} \geq \frac{2Z_0}{t_0} + \frac{½\lambda^* e^{\omega^*}\varphi^* z_0^2}{t_0^2} \tag{1.4.21}$$

i.e. if t_0 is sufficiently large.

Further, and again as $z_0 \geq 0,$ we see that the second requirement for Φ_0 , the lower solution to $\hat{u}$, inequality (1.4.12), will be satisfied if

$$\frac{\mu_0\varphi^*}{t_0} \leq \omega^* - u_0 \text{ for } x\epsilon\Omega,$$

i.e. if $t_0 \geq \mu_0 \sup_{x\in\Omega}\left[\frac{\varphi^*}{\omega^*-u_0}\right].$ (1.4.22)

We conclude, therefore, that if μ_0 is defined by equation (1.4.15) and t_0 is chosen 'large enough' to satisfy both inequalities (1.4.21) and (1.4.22), then $\Phi_0(x,t)$ as defined by (1.4.4) will be a lower solution to $\hat{u}$ for all $t>0$. .

We note, however, that provided

$$\sup_{x\in\Omega}\left[\frac{\varphi^*}{\omega^*-u_0}\right] \text{ is not large,}$$

inequalities (1.4.21) and (1.4.22) only require that t_0 be chosen of order one.

Having established a lower solution to $\hat{u}$ which exists for all $t>0$ we are motivated by the form of inequality (1.4.14) to look for a second lower solution to $\hat{u}$ which will exist for suitably large t, but in which we may choose μ_0 much closer to the maximum value allowed by condition (1.4.13).

If this second function is labelled $\Phi_1(x,t)$,

where

$$\Phi_1(x,t) = \frac{\mu_1\varphi^*}{(t+t_1)} - \frac{z_1(x)}{(t+t_1)^2} \tag{1.4.23}$$

then we may repeat the arguments applied to Φ_0 to reach the conclusion that

$\Phi_1(x,t)$ will also be a lower solution to $\hat{u}$ for all $t \geq \bar{t}$ if

$$\mu_1\varphi^*[1 - \mu_1 \tfrac{1}{2}\lambda^* I_3] \geq \frac{2z_1}{(t+t_1)}[1 - \tfrac{1}{2}\mu_1\lambda^* e^{\omega^*}\varphi^*] + \tfrac{1}{2}\lambda^* e^{\omega^*}\frac{z_1^2}{(t+t_1)^2}$$

$$\text{for } x \in \Omega,\ t \geq \bar{t} \tag{1.4.24}$$

from (1.4.14). In addition, we must be able to verify that

$$\Phi_1(x,t) = \frac{\mu_1\varphi^*}{(\bar{t}+t_1)} - \frac{z_1(x)}{(\bar{t}+t_1)^2} \leq \hat{u}(x,\bar{t}) \text{ for } x \text{ in } \Omega. \tag{1.4.25}$$

As we now look to choose μ_1 much closer to the maximum value allowable by

condition (1.4.13) we set

$$\mu_1 = [\tfrac{1}{2}\lambda^* I_3]^{-1} - \delta > 0 \tag{1.4.26}$$

for some $\delta > 0$ (and hopefully small) yet to be determined.

With this μ_1, we see from equation (1.4.10) (with $e_0 - e_1$, and $\mu_0 - \mu_1$) that

$$e_1(x) = \mu_1\varphi^*[1 - \mu_1\tfrac{1}{2}\lambda^* I_3] = \delta\varphi^*[(\tfrac{1}{2}\lambda^* I_3)^{-1} - \delta]. \tag{1.4.27}$$

From equations (1.4.17), (1.4.18), with $z_0 - z_1$ and $\mu_0 - \mu_1$, we again conclude

that an estimate of the form (1.4.19) remains valid for z_1 and hence that

$$z_1(x) \leq Z_1\varphi^* \text{ for } x \text{ in } \Omega \tag{1.4.28}$$

and some positive constant z_1.

To establish Φ_1 as a lower solution for $\hat{u}$ we must satisfy inequality (1.4.24).

For $\mu_1 z_1 \geq 0$ inequality (1.4.24) will hold for all $x \in \Omega, t \geq \bar{t}$ if

$$\varphi^* \mu_1 [1 - \mu_1 \tfrac{1}{2} \lambda^* I_3] \geq \frac{2 z_1}{(\bar{t} + t_1)} + \frac{\tfrac{1}{2} \lambda^* e^{\omega^*} z_1^2}{(\bar{t} + t_1)^2} \quad \text{for } x \in \Omega.$$

On substituting for μ_1 by (1.4.26) and estimating $z_1(x)$ by (1.4.28) we see that this is automatic if

$$\delta [(\tfrac{1}{2} \lambda^* I_3)^{-1} - \delta] \geq \frac{2 Z_1}{(\bar{t} + t_1)} + \frac{\tfrac{1}{2} \lambda^* e^{\omega^*} \varphi^* z_1^2}{(\bar{t} + t_1)^2} \quad \text{for } x \in \Omega. \tag{1.4.29}$$

For small δ, inequality (1.4.29) requires that

$$(\bar{t} + t_1) > Z_1' \delta^{-1} \tag{1.4.30}$$

for positive constant Z_1'.

We must also ensure satisfaction of inequality (1.4.25), i.e. that

$$\Phi_1(x, \bar{t}) - \frac{\mu_1 \varphi^*}{(\bar{t} + t_1)} - \frac{z_1(x)}{(\bar{t} + t_1)^2} \leq \hat{u}(x, \bar{t}) \quad \text{for } x \in \Omega$$

and make use of our information on Φ_0.

We have previously established that $\Phi_0(x,t)$ is a lower solution to $\hat{u}$ for all $t>0$ (and hence at $t-\bar{t}$) provided μ_0 and t_0 are chosen suitably. It follows that if

$$\Phi_1(x,\bar{t}) \le \Phi_0(x,\bar{t}) \quad \text{for} \quad x\in\Omega$$

i.e. if

$$\frac{\mu_1\varphi^*}{(\bar{t}+t_1)} - \frac{z_1(x)}{(\bar{t}+t_1)^2} \le \frac{\mu_0\varphi^*}{(\bar{t}+t_0)} - \frac{z_0(x)}{(\bar{t}+t_0)^2} \quad \text{for} \quad x\in\Omega \tag{1.4.31}$$

then condition (1.4.25) will hold as required. If $\bar{t}$ is large, then as z_0 satisfies (1.4.17), $z_1\ge 0,$ and μ_0 is not 'small', inequality (1.4.31) requires that

$$\frac{\mu_1}{(\bar{t}+t_1)} \le \frac{\mu_0'}{(\bar{t}+t_0)} \quad \text{for} \quad \mu_0' - \mu_0 - \frac{Z_0}{(\bar{t}+t_0)}$$

which is satisfied if

$$\mu_1 \le \mu_0'\left\{\frac{\bar{t}+t_1}{\bar{t}+t_0}\right\}. \tag{1.4.32}$$

We consequently choose t_1 such that

$$t_1 - t_0+c(\bar{t}+t_0) \tag{1.4.33}$$

in which case (1.4.32) reduces to

$$\mu_1 \leq \mu_0'(1 + c)\,. \tag{1.4.34}$$

With this choice of t_1, inequality (1.4.30) requires that

$$(1+c)\,(\overline{t}+t_0) \geq z_1'\delta^{-1}$$

which is satisfied if

$$\overline{t} \geq z\delta^{-1}, \quad \text{where} \quad z = \frac{z_1'}{(1+c)}\,. \tag{1.4.35}$$

We conclude, therefore, that condition (1.4.34) is satisfied provided c is a suitably large constant and hence that $\Phi_1(x,t)$, as defined by (1.4.23), is, with μ_1 as in (1.4.26), a lower solution to $\hat{u}$ for all $t \geq \overline{t}$ where $\overline{t}$ satisfies (1.4.35).

To summarise, we find that $\Phi_1(x,t)$ where

$$\Phi_1(x,t) = \frac{\mu_1\varphi^*}{(t+t_1)} - \frac{z_1(x)}{(t+t_1)^2}$$

with

$$\mu_1 = [\tfrac{1}{2}\lambda^* I_3]^{-1} - \delta > 0$$

is a lower solution to $\hat{u} = \omega^* - u^*$ for all $t \geq \overline{t}$ where $\overline{t} \geq z\delta^{-1}$ and $t_1 = t_0 + c(\overline{t}+t_0)$, with t_0, c and z suitably defined positive constants.

It remains, however, to establish the required upper solution to u . Recalling the analysis of Section 1.2.2, we see that, if the function $u_1(x,t)$ is defined (as in (1.2.37)) as

$$u_1(x,t) = u(x,t) - u^*(x,t),$$

then $\psi_1(x,t)$, as defined by (1.2.48), is an upper solution to u_1 provided $u<\omega^*$. It was also established in Section 1.2.2 that provided condition (1.2.53) holds i.e. if

$$t \le t_1(\lambda-\lambda^*)^{-1/2},$$

for some positive t_1, then $u<\omega^*$ as required. It follows therefore that if

$$Z\delta^{-1} \le t \le t_1(\lambda-\lambda^*)^{-1/2} \tag{1.4.36}$$

for positive $Z,\ t_1$ and small $\delta>0$, then $\Phi_1(x,t)$ is a lower solution to $\hat{u}=\omega^*-u^*$, and $\psi_1(x,t)$ as defined in (1.2.48) as

$$\psi_1(x,t) = (\lambda-\lambda^*)(I_1\varphi^*t + \Phi^*)$$

is an upper solution to $u_1 = u - u^*$.

Hence,

$$u(x,t) = u^* + u_1 \le \omega^* - \Phi_1 + \psi_1 \quad \text{for} \quad x\in\Omega, \quad t \text{ satisfying (1.4.36),}$$

so that $$u(x,t) \le \omega^*(x) - \frac{\mu_1\varphi^*(x)}{(t+t_1)} + \frac{z_1(x)}{(t+t_1)^2} + (\lambda-\lambda^*)\left[I_1\varphi^* t + \Phi^*\right] \tag{1.4.37}$$

where

$$\mu_1(\tfrac{1}{2}\lambda^* I_3)^{-1} - \delta > 0$$

with

$$t_1 = t_0 + c(\bar{t} + t_0)$$

and

$$\bar{t} = Z\delta^{-1}.$$

1.4.2. An upper solution in time region II

Section 1.4.1 has established an upper solution to u which, like that of Section 1.2.2 exists while $u<\omega^*$ but which approaches a maximal value as t becomes large.

As in Section 1.3.2, however, we again note that at some particular time t prior to the failure of this upper solution, we may estimate u as in (1.3.53) i.e.

$$\omega^* - C_1\varphi^*(\lambda-\lambda^*)^{1/2} \le u(x,t) \le \omega^* - C_2\varphi^*(\lambda-\lambda^*)^{1/2} \qquad (1.4.53)$$

for some C_1 and C_2.

The purpose of this section is to identify a further upper solution to u which exists for all t of the order of $(\lambda-\lambda^*)^{-1/2}$ and with which it will be possible to derive an upper estimate for u at times when $u>\omega^*$.

If we call this function $\overline{u}(x,t)$ then $\overline{u}(x,t)$ will be an upper solution to u for $t \ge \overline{t}_0\,(=\overline{\tau}_0(\lambda-\lambda^*)^{-1/2})$ if

$$\overline{u}_t \ge \nabla^2\overline{u} + \lambda e^{\overline{u}} \quad \text{in} \quad \Omega,\, t \ge \overline{t}_0, \qquad (1.4.38)$$

$$\overline{u}(x,\overline{t}_0) \ge u(x,\overline{t}_0) \quad \text{in} \quad \Omega, \qquad (1.4.39)$$

and $\quad \dfrac{\partial \bar{u}}{\partial n} + \beta \bar{u} \geq 0 \qquad$ on $\partial\Omega, t \geq \bar{t}_0$. $\qquad$ (1.4.40)

As in Section 1.3.2 we are again motivated by the estimates (1.3.53) to look for $\bar{u}(x,t)$ defined as

$$\bar{u}(x,t) = \omega^*(x) + \epsilon^{1/2}\bar{u}_1(x,t) + \epsilon\bar{u}_2(x,t) \tag{1.4.41}$$

where $\epsilon = (\lambda-\lambda^*)$ is assumed small and the functions $\bar{u}_1$ and $\bar{u}_2$ are to be determined. If we again rescale t in terms of the order one variable τ defined in (1.3.60) as

$$\tau = (\lambda-\lambda^*)^{1/2}t = \epsilon^{1/2}t$$

and substitute for $\bar{u}$ in inequality (1.4.38) then $\bar{u}$ must be chosen to satisfy

$$\epsilon\bar{u}_{1\tau}+\epsilon^{3/2}\bar{u}_{2\tau} - (\nabla^2\omega^* + \epsilon^{1/2}\nabla^2\bar{u}_1 + \epsilon\nabla^2\bar{u}_2)-(\lambda^* + \epsilon)\, e^{\omega^*}\exp(\epsilon^{1/2}\bar{u}_1 + \epsilon\bar{u}_2) \geq 0$$

for $x\in\Omega\, t \geq \bar{t}_0 \qquad$ (1.4.42)

where $t_0 = \bar{\tau}_0\epsilon^{-1/2}$.

We proceed along the route established in Section 1.3.2 and again find that inequality (1.4.42) will be satisfied (by equality) up to terms of order $\epsilon^{1/2}$ if we choose

$$\bar{u}_1(x,t) = \bar{a}(\tau)\,\varphi^*(x) \tag{1.4.43}$$

for some function $\bar{a}(\tau)$ and φ^* the solution (1.2.3), (1.2.4) satisfying (1.2.34).

The highest order terms remaining in (1.4.42) are now of order ϵ, and satisfaction of the inequality at this level requires that

$$\bar{u}_{1\tau} - \left\{\nabla^2\bar{u}_2 + e^{\omega^*} + \lambda^* e^{\omega^*}\left[\bar{u}_2 + \tfrac{1}{2}\bar{u}_1^2\right]\right\} \geq 0$$

i.e. that

$$\bar{a}_\tau\varphi^* - \left\{\nabla^2\bar{u}_2 + e^{\omega^*} + \lambda^* e^{\omega^*}\left[\bar{u}_2 + \tfrac{1}{2}\bar{a}^2\varphi^{*2}\right]\right\} \geq 0 \quad \text{for} \quad x\in\Omega, \tau \geq \bar{\tau}_0. \tag{1.4.44}$$

Hence, we must choose $\bar{u}_2$ such that

$$\nabla^2\bar{u}_2 + \lambda^* e^{\omega^*}\bar{u}_2 \leq \bar{a}_\tau\varphi^* - \left(e^{\omega^*} + \tfrac{1}{2}\lambda^* e^{\omega^*}\bar{a}^2\varphi^{*2}\right) \quad \text{for} \quad x \quad \text{in} \quad \Omega, \tau \geq \bar{\tau}_0. \tag{1.4.45}$$

On multiplying both sides of inequality (1.4.45) by φ^* and integrating over Ω we find that $\bar{a}(\tau)$ must satisfy

$$\bar{a}_\tau \geq I_1 + \tfrac{1}{2}\lambda^* I_3\bar{a}^2. \tag{1.4.46}$$

To match with the analysis of Section 1.4.1 we set

$$\bar{a}(\tau) = \bar{C}_1\tan\left[\bar{C}_2\tau - \frac{\pi}{2}\right] \tag{1.4.47}$$

for $\bar{C}_1$ and $\bar{C}_2$ chosen to ensure inequality (1.4.46) is satisfied, i.e. such that

$$\bar{a}_\tau = \bar{C}_1\bar{C}_2 + \frac{\bar{C}_2}{\bar{C}_1}a^2 \geq I_1 + \tfrac{1}{2}\lambda^* I_3\bar{a}^2, \quad \text{for} \quad \tau \geq \bar{\tau}_0. \tag{1.4.48}$$

Clearly, therefore if inequality (1.4.48) is to hold for all possible $\bar{a}$ we must have that

$$\bar{C}_1\bar{C}_2 \geq I_1 \quad \text{and} \quad \frac{\bar{C}_2}{\bar{C}_1} \geq \tfrac{1}{2}\lambda^* I_3. \tag{1.4.49}$$

With this choice of $\bar{a}$, inequality (1.4.45) becomes

$$\nabla^2\bar{u}_2 + \lambda^* e^{\omega^*}\bar{u}_2 \leq \bar{a}_\tau\varphi^* - (e^{\omega^*} + \tfrac{1}{2}\lambda^* e^{\omega^*}\bar{a}^2\varphi^{*2})$$
$$- \left[\bar{C}_1\bar{C}_2\varphi^* - e^{\omega^*}\right] + \left[\frac{\bar{C}_2}{\bar{C}_1} - \tfrac{1}{2}\lambda^* e^{\omega^*}\varphi^*\right]\bar{a}^2\varphi^*$$

$$\text{for} \quad x\in\Omega, \tau \geq \bar{\tau}_0. \tag{1.4.50}$$

We consequently choose $\bar{u}_2$ such that

$$\nabla^2\bar{u}_2 + \lambda^* e^{\omega^*}\bar{u}_2 = \left[\bar{C}_1\bar{C}_2\varphi^* - e^{\omega^*}\right] + \left[\frac{\bar{C}_2}{\bar{C}_1} - \tfrac{1}{2}\lambda^* e^{\omega^*}\varphi^*\right]\bar{a}^2\varphi^* - e(x,\tau) \quad \text{for} \quad x\in\Omega,\ \tau \geq \bar{\tau}_0 \tag{1.4.51}$$

with

$$\frac{\partial\bar{u}_2}{\partial n} + \beta\bar{u}_2 = 0 \quad \text{on} \quad \partial\Omega, \tau \geq \bar{\tau}_0 \tag{1.4.52}$$

for some function $e(x,\tau) \geq 0$ yet to be determined. Hence, on multiplying through (1.4.51) by φ^* and integrating over Ω we see that

$$(\overline{C}_1\overline{C}_2-I_1) + \left(\frac{\overline{C}_2}{\overline{C}_1} - ½\lambda^* I_3\right)\overline{a}^2 - \int_\Omega e(x,\tau)\varphi^* = 0 \quad \text{for} \quad \tau\geq\overline{\tau}_0$$

which is satisfied if

$$e(x,\tau) = e(\tau)\varphi^*(x) \quad \text{for} \quad x\in\Omega, \tau \geq \overline{\tau}_0, \tag{1.4.53}$$

and
$$e(\tau) = (\overline{C}_1\overline{C}_2 - I_1) + \left(\frac{\overline{C}_2}{\overline{C}_1} - ½\lambda^* I_3\right)\overline{a}^2(\tau) \quad \text{for} \quad \tau \geq \overline{\tau}_0. \tag{1.4.54}$$

With this choice of the functions $\overline{u}_1$ and $\overline{u}_2$, the order ϵ terms of (1.4.42) (as described by the left hand side of (1.4.44)) are

$$\epsilon e(\tau)\varphi^*$$

and are greater than or equal to zero (in light of (1.4.49) as required.

On substituting for $\overline{u}_1$ and $\overline{u}_2$ in the 'full' inequality (1.4.42) we now see that, for $\overline{u}$ to be an upper solution to u for $\tau \geq \overline{\tau}_0$, we must ensure that

$$\begin{aligned}\epsilon e(\tau)\varphi^* + \epsilon^{3/2}\,\overline{u}_{2\tau} - \lambda^* e^{\omega^*}\{\exp(\epsilon^{½}\overline{a}\varphi^* + \epsilon\overline{u}_2) \\ - (1+\epsilon^{½}\overline{a}\varphi^* + ½\overline{a}^2\varphi^{*2}+\epsilon\overline{u}_2)\} \\ - \epsilon e^{\omega^*}\{\exp(\epsilon^{½}\overline{a}\varphi^* + \epsilon\overline{u}_2) - 1\} \geq 0 \\ \text{for} \quad x\in\Omega, \tau \geq \overline{\tau}_0.\end{aligned} \tag{1.4.55}$$

Following some manipulation, inequality (1.4.55) above is seen to be equivalent to

$$\epsilon e(\tau)\varphi^* + \epsilon^{3/2}\,\bar{u}_{2\tau}$$
$$- \lambda^* e^{\omega^*}\{\exp(\epsilon^{1/2}\bar{a}\varphi^*) - (1 + \epsilon^{1/2}\bar{a}\varphi^* + 1/2\epsilon\bar{a}^2\varphi^{*2})$$
$$+ [\exp[\epsilon^{1/2}\bar{a}\varphi^*] - 1][\exp(\epsilon\bar{u}_2) - 1] \tag{1.4.56}$$
$$+ \exp(\epsilon\bar{u}_2) - (1 + \epsilon\bar{u}_2)\}$$
$$-\epsilon e^{\omega^*}\{\exp(\epsilon^{1/2}\bar{a}\varphi^* + \epsilon\bar{u}_2) - 1\} \ge 0$$

which will be satisfied for all possible $a(\tau)$ and $\bar{u}_2$ if

$$\epsilon e(\tau)\varphi^* - \epsilon^{3/2}\,|\,\bar{u}_{2\tau}\,|$$
$$-\lambda^* e^{\omega^*}\{\exp(\epsilon^{1/2}\,|\,\bar{a}\,|\,\varphi^*) - (1 + \epsilon^{1/2}\,|\,\bar{a}\,|\,\varphi^* + 1/2\epsilon\bar{a}^2\varphi^{*2})$$
$$+[\exp[\epsilon^{1/2}\,|\bar{a}\,|\,\varphi^*] - 1]\,[\exp(\epsilon\,|\,\bar{u}_2\,|) - 1]$$
$$+\exp(\epsilon\,|\,\bar{u}_2\,|) - (1 + \epsilon\,|\,\bar{u}_2\,|)\}$$
$$-\,\epsilon e^{\omega^*}\{\exp(\epsilon^{1/2}\,|\,\bar{a}\,|\,\varphi^* + \epsilon\,|\,\bar{u}_2\,|) - 1\} \ge 0$$

for $x \in \Omega, \tau \ge \bar{\tau}_0$. (1.4.57)

Hence, if we assume (as in Section 1.3.2) that

$$\epsilon^{1/2}\,|\bar{a}|\,\varphi^* < 1 \quad \text{and} \quad \epsilon\,|\bar{u}_2| < 1 \tag{1.4.58}$$

then the inequalities (1.3.87) remain valid when applied to $\bar{a}$ and $\bar{u}_2$,

i.e. there exist positive constants, say in this case $\bar{\mathbf{M}}_1$ and $\bar{\mathbf{M}}_2$,

such that

$$\exp\,(\epsilon^{1/2}\,|\bar{a}|\,\varphi^*) \le 1 + \bar{\mathbf{M}}_1\epsilon^{1/2}\,|\bar{a}|\,\varphi^* \quad \text{and} \quad \exp\,(\epsilon\,|\bar{u}_2|\,) \le 1 + \bar{\mathbf{M}}_2\epsilon\,|\bar{u}_2|. \tag{1.4.59}$$

In addition, there also exists positive $\bar{\mathbf{M}}_3$ and $\bar{\mathbf{M}}_4$, say, for which

$$\exp\,(\epsilon\,|\bar{u}_2|\,) \le 1 + \epsilon\,|\bar{u}_2| + \mathbf{M}_3\,1/2\,\epsilon^2\,|\bar{u}_2|^2 \tag{1.4.60}$$

and

$$\exp\,(\epsilon^{1/2}\,|\bar{a}|\,\varphi^*) - (1 + \epsilon^{1/2}|\bar{a}|\,\varphi^* + 1/2\epsilon\,|\bar{a}|^2\,\varphi^{*2}) \le \bar{\mathbf{M}}_4\,\epsilon^{3/2}\,|\bar{a}|^3\,\varphi^{*3} \tag{1.4.61}$$

e.g. $\bar{M}_3 = 1 + 2\epsilon|\bar{u}_2| \exp(\epsilon|\bar{u}_2|)$ and $\bar{M}_4 \geq 6 \exp(\epsilon^{1/2}|\bar{a}|\varphi^*)$.

Hence if the inequalities (1.4.59), (1.4.60) and (1.4.61) are used to estimate the appropriate terms in the left hand side of (1.4.57), then this condition will be satisfied for all x in Ω, $\tau \geq \tau_0$ and $\bar{a}$, $\bar{u}_2$ satisfying (1.4.58) if

$$\begin{aligned} &\epsilon e(\tau)\varphi^* - \epsilon^{3/2}|\bar{u}_{2\tau}| - \lambda^* e^{\omega^*}\{M_4\epsilon^{3/2}|\bar{a}|^3\varphi^{*3} \\ &\qquad + M_1M_2\epsilon^{3/2}|\bar{a}||\bar{u}_2|\varphi^* + \tfrac{1}{2}M_3\epsilon^2|\bar{u}_2|^2\} \\ &\quad - \epsilon e^{\omega^*}\{M_1\epsilon^{1/2}|\bar{a}|\varphi^* + M_2\epsilon|\bar{u}_2| + M_1M_2\epsilon^{3/2}|\bar{a}||\bar{u}_2|\varphi^*\} \geq 0. \end{aligned} \tag{1.4.62}$$

Before proceeding, we return to equations (1.4.51), (1.4.52) in order to estimate the 'size' of $\bar{u}_2$.

On substituting for $e(x,\tau)$ by equations (1.4.53), (1.4.54), we see from (1.4.51), (1.4.52) that $\bar{u}_2$ satisfies

$$\nabla^2\bar{u}_2 + \lambda^* e^{\omega^*}\bar{u}_2 = \left[I_1\varphi^* - e^{\omega^*}\right] + \tfrac{1}{2}\lambda^*\varphi^*\bar{a}^2\left[I_3 - \varphi^* e^{\omega^*}\right]$$

$$\text{in } \Omega,\ \tau \geq \bar{\tau}_0 \tag{1.4.63}$$

with $$\frac{\partial\bar{u}_2}{\partial n} + \beta\bar{u}_2 = 0 \quad \text{on } \partial\Omega,\ \tau \geq \bar{\tau}_0 \tag{1.4.64}$$

and hence that $\bar{u}_2$ may be chosen such that

$$|\overline{u}_2(x,\tau)| \le \overline{E}_1\, \varphi^*[1+\overline{a}^2] \quad \text{for} \quad x \text{ in } \Omega,\ \tau \ge \overline{\tau}_0, \tag{1.4.65}$$

for some positive constant $\overline{E}_1$.

Further, on differentiating (1.4.63), (1.4.64) with respect to τ, it follows that

$$\nabla^2 \overline{u}_{2\tau} + \lambda^* e^{\omega^*}\overline{u}_{2\tau} - \lambda^*\varphi^*\overline{a}\left[I_3 - \varphi^* e^{\omega^*}\right]\left[\overline{C}_1\overline{C}_2 + \frac{\overline{C}_2}{\overline{C}_1}\overline{a}^2\right] \text{ in } \Omega,\ \tau \ge \overline{\tau}_0,$$

with $\dfrac{\partial\overline{u}_{2\tau}}{\partial n} - \beta\overline{u}_{2\tau} + 0$ on $\partial\Omega,\ t \ge \overline{\tau}_0,$

and hence that there exists a positive constant $\overline{E}_2$ for which

$$|\overline{u}_{2\tau}(x,\ \tau)| \le \overline{E}_2\varphi^*[1+|\overline{a}|^3] \quad \text{for} \quad x \text{ in } \Omega, \tau \ge \tau_0. \tag{1.4.66}$$

Clearly, (1.4.65) allows the condition (1.4.58) to be simplified to

$$\epsilon^{1/2}|\overline{a}| < c \quad \text{for some positive } c. \tag{1.4.67}$$

Applying the estimates (1.4.65) and (1.4.66) in (1.4.61) yields a condition which we seek to satisfy in order for $\overline{u}(x,t)$ to be an upper solution to u for $\tau \ge \tau_0$;

$$\epsilon e(\tau)\varphi^* \geq \epsilon^{3/2}\,\bar{E}_2\varphi^*[1+|\bar{a}|^3]$$

$$+\lambda^* e^{\omega^*}\{\epsilon^{3/2}\,\bar{\mathbf{M}}_4\,|\,\bar{a}\,|^3\varphi^{*3} + \bar{\mathbf{M}}_1\bar{\mathbf{M}}_2\epsilon^{3/2}\,|\,\bar{a}\,|\,\varphi^{*2}\,[1+\bar{a}^2]\,\bar{E}_1$$

$$+ \tfrac{1}{2}\,\bar{\mathbf{M}}_3\,\epsilon^2\,E_1^2\,\varphi^{*2}[1+\bar{a}^2]^2\}$$

$$+\epsilon e^{\omega^*}\{\epsilon^{1/2}\bar{\mathbf{M}}_1\,|\,\bar{a}\,|\,\varphi^* + \epsilon\bar{\mathbf{M}}_2\bar{E}_1\,[1+\bar{a}^2] + \epsilon^{3/2}\mathbf{M}_1\mathbf{M}_2|\bar{a}|\varphi^{*2}\bar{E}_1[+\,\bar{a}^2]\}$$

$$\text{for} \quad x\in\Omega,\ \tau \geq \bar{\tau}_0. \tag{1.4.68}$$

It remains to choose a function $\bar{a}(\tau)$ which will ensure that this inequality is satisfied within the given time range, i.e. we must choose the constants $\bar{C}_1$ and $\bar{C}_2$.

To ensure that the necessary conditions (1.4.49) are satisfied, we set

$$\bar{C}_1\bar{C}_2 = I_1 + \bar{K}_1, \quad \text{and} \quad \frac{\bar{C}_2}{\bar{C}_1} = \tfrac{1}{2}\lambda^* I_3 + \bar{K}_2 \tag{1.4.69}$$

for some positive constants $\bar{K}_1$, $\bar{K}_2$.

Hence,

$$\bar{C}_1 = \left[\frac{I_1 + \bar{K}_1}{\tfrac{1}{2}\lambda^* I_3 + \bar{K}_2}\right]^{1/2} \tag{1.4.70}$$

and

and

$$\overline{C}_2 = \{[\tfrac{1}{2}\lambda^* I_3 + \overline{K}_2]\,[I_1 + \overline{K}_1]\}^{\frac{1}{2}} \,. \tag{1.4.71}$$

Further, as the size of $\overline{u}\,(x,\tau)$ is dominated by the function $\overline{a}(\tau)$, and as

$\overline{a}(\tau)$ becomes infinitely large as τ approaches $\overline{\tau}_b$ where

$$\overline{\tau}_b = \frac{\pi}{\overline{C}_2} = \frac{\pi}{\{[\tfrac{1}{2}\lambda^* I_3 + \overline{K}_2][I_1 + \overline{K}_1]\}^{\frac{1}{2}}} \tag{1.4.72}$$

it is clear that it is the value of $max(\overline{K}_1,\overline{K}_2)$ which will determine how large $\overline{\tau}_b$ can be.

Returning to inequality (1.4.68), we see that, for $e(\tau)$ as defined by equation (1.4.54), and $\overline{C}_1, \overline{C}_2$ as in (1.4.69), then

$$e(\tau) = \overline{K}_1 + \overline{K}_2\overline{a}^2(\tau)\,,$$

and (1.4.68) requires that

$$\begin{aligned}\epsilon\varphi^*[\overline{K}_1 + \overline{K}_2\,|\,\overline{a}\,|^2] \geq\ & \epsilon^{3/2}\overline{E}_2\varphi^*[1 + |\overline{a}|^3] \\ & +\lambda^* e^{\omega^*}\{\,\overline{\mathbf{M}}_4\epsilon^{3/2}|\overline{a}|\varphi^{*3} + \overline{\mathbf{M}}_1\overline{\mathbf{M}}_2\overline{E}_1\epsilon^{3/2}\,|\overline{a}|\,\varphi^{*2}\,[1+\overline{a}^2] \\ & \qquad +\tfrac{1}{2}\overline{\mathbf{M}}_3\epsilon^2\overline{E}_1^2\varphi^{*2}\,[1+\overline{a}^2]^2\,\} \\ & +\epsilon e^{\omega^*}\{\epsilon^{\frac{1}{2}}\overline{\mathbf{M}}_1\,|\,\overline{a}\,|\,\varphi^* + \epsilon\overline{\mathbf{M}}_2\overline{E}_1\,[1+\overline{a}^2] + \epsilon^{3/2}\overline{\mathbf{M}}_1\overline{\mathbf{M}}_2\,|\,\overline{a}\,|\,\varphi^{*2}\,\overline{E}_1\,[1+\overline{a}^2]\} \\ & \qquad \text{for } x \text{ in } \Omega,\, \tau \geq \overline{\tau}_0 .\end{aligned} \tag{1.4.73}$$

Clearly, there exist positive constants $\bar{A}_1, \bar{A}_2,$ and $\bar{A}_3$ for which the right hand side of inequality (1.4.73) is no greater than

$$\epsilon^{3/2}\varphi^*\{\ \bar{A}_1|\bar{a}|^3 + \bar{A}_2|\bar{a}| + \bar{A}_3\ \}\ .$$

Hence, if

$$\bar{A} = \bar{A}_1 + \bar{A}_2 + \bar{A}_3,$$

then inequality (1.4.73) will be satisfied for all $\bar{a}$ if

$$\text{and}\qquad \begin{array}{l} \bar{K}_1 + \bar{K}_2\bar{a}^2 \geq \epsilon^{1/2}\bar{A}\ |\bar{a}|^3 \\ \bar{K}_1 + \bar{K}_2\bar{a}^2 \geq \epsilon^{1/2}\bar{A} \end{array} \qquad \text{when} \quad \begin{array}{l} |\bar{a}| \geq 1 \\ |\bar{a}| < 1. \end{array} \tag{1.4.74}$$

In line with (1.4.67), we allow $|\bar{a}|$ to take the value

$$|\bar{a}| = \epsilon^{-\gamma} \qquad\qquad \text{for some} \quad 0 < \gamma < 1/2\ , \tag{1.4.75}$$

in which case, the first inequality of (1.4.74) requires that

$$\bar{K}_1 + \epsilon^{-2\gamma}\bar{K}_2 \geq \epsilon^{1/2-3\gamma}\bar{A}\ .$$

To satisfy the second inequality of (1.4.74) for all $|\bar{a}| < 1$, we must also choose

$$\bar{K}_1 \geq \epsilon^{1/2}\bar{A}\ . \tag{1.4.76}$$

In order to derive a 'best' estimate for the blow up time for u , however, we would like to choose both $\bar{K}_1$ and $\bar{K}_2$ as small as possible, i.e. so that $\max(\bar{K}_1, \bar{K}_2)$ is minimised. At this point, minimum values of $\bar{K}_1$ and $\bar{K}_2$ must satisfy

$$\bar{K}_1 \geq \epsilon^{1/2}\bar{A} \text{ , and } \bar{K}_2 \geq \epsilon^{1/2-\gamma}\bar{A} - \epsilon^{2\gamma}\bar{K}_1 - \epsilon^{1/2-\gamma}\bar{A}' \text{ say,} \tag{1.4.77}$$

for some positive $\bar{A}, \bar{A}'$ and γ as in (1.4.75).

We now consider the other requirements necessary for $\bar{u}$ to be an upper solution to u , i.e. inequalities (1.4.39) and (1.4.40). The boundary condition (1.4.40) requires that

$$\frac{\partial \bar{u}}{\partial n} + \beta \bar{u} \geq 0 \qquad \text{on } \partial\Omega, \tau \geq \bar{\tau}_0$$

and is automatic in light of the definitions of $\bar{u}_1$ and $\bar{u}_2$, i.e. from (1.2.4) and (1.4.52).

The initial condition (1.4.39), however, requires that

$$\bar{u}(x, \bar{\tau}_0) \geq u(x, \bar{\tau}_0) \qquad \text{for } x \text{ in } \Omega$$

and must be investigated in more detail.

We begin by evaluating $\bar{u}(x, \bar{\tau}_0)$.

Section 1.3 has established that, as $\bar{\tau}_0 - \bar{\tau}_0 \epsilon^{-1/2}$,

then

$$u(x,\bar{\tau}_0) \geq \omega^* - \frac{C\epsilon^{1/2}\varphi^*}{\bar{\tau}_0}$$

for some positive c (from (1.3.45) for example).

Hence as

$$\bar{a}(\bar{\tau}_0) - \bar{C}_1 \tan\left[\, \bar{C}_2\bar{\tau}_0 - \pi/2 \,\right],$$

from (1.4.47), and

$$\begin{aligned} \bar{u}(x,\bar{\tau}_0) &- \omega^*(x) + \epsilon^{1/2}\varphi^*(x)\,\bar{a}(\bar{\tau}_0) + \epsilon\bar{u}_2(x,\bar{\tau}_0) \\ &\geq \omega^* + \epsilon^{1/2}\varphi^*\bar{a}(\bar{\tau}_0) - \epsilon\varphi^*\bar{E}_1\left[1+\bar{a}^2(\bar{\tau}_0)\right] \end{aligned} \qquad (1.4.78)$$

using the estimate (1.4.65), we anticipate that sufficiently large $\bar{\tau}_0$ will ensure the initial condition (1.4.39) is satisfied as required.

However, it is also clear that the upper solution which is ‘closest’ to u will require that $\bar{\tau}_0$ be chosen ‘as small as possible’.

If $\bar{\tau}_0$ is ‘small’, then from (1.4.47) we see that

$$\bar{a}\,(\bar{\tau}_0) \geq -\frac{\bar{C}_1}{\bar{C}_2\bar{\tau}_0}$$

and

$$\bar{a}^2(\bar{\tau}_0) \le \left[\frac{\bar{C}_1}{\bar{C}_2\bar{\tau}_0}\right]^2 .$$

Hence, as

$$\frac{\bar{C}_2}{\bar{C}_1} = \tfrac{1}{2}\lambda^* I_3 + \bar{K}_2$$

from (1.4.69), then

and
$$\left.\begin{aligned} \bar{a}(\bar{\tau}_0) &\ge \frac{-1}{(\tfrac{1}{2}\lambda^* I_3 + \bar{K}_2)\bar{\tau}_0} \\ \bar{a}^2(\bar{\tau}_0) &\le \frac{1}{(\tfrac{1}{2}\lambda^* I_3 + \bar{K}_2)^2\bar{\tau}_0^2} \end{aligned}\right\} \tag{1.4.79}$$

Inequality (1.4.78) and the estimates (1.4.79) combine to establish that

$$\begin{aligned} \bar{u}(x,\bar{\tau}_0) &\ge \omega^*(x) + \epsilon^{1/2}\bar{a}(\bar{\tau}_0)\varphi^*(x) - \epsilon\bar{E}_1\varphi^*(x)\left[1+\bar{a}^2(\bar{\tau}_0)\right] \\ &\ge \omega^* - \frac{\epsilon^{1/2}\varphi^*}{[\tfrac{1}{2}\lambda^* I_3 + \bar{K}_2]\bar{\tau}_0} - \epsilon\bar{E}_1\varphi^*\left\{1+\frac{1}{[\tfrac{1}{2}\lambda^* I_3 + \bar{K}_2]^2\bar{\tau}_0^2}\right\}. \end{aligned} \tag{1.4.80}$$

Further, the condition (1.4.67) requires that

$$|\bar{a}(\tau)| < \epsilon^{-\gamma}$$

for some $0<\gamma<\tfrac{1}{2}$ and all τ in the considered range, which suggests that

$$|\bar{a}(\bar{\tau}_0)| \le \frac{1}{[\tfrac{1}{2}\lambda^* I_3 + \bar{K}_2]\bar{\tau}_0} \le \epsilon^{-\gamma} .$$

We consequently choose $\bar{\tau}_0$ such that

$$\bar{\tau}_0 \ge \bar{C}\,\epsilon^{\gamma} \tag{1.4.81}$$

for suitably large $\bar{\mathcal{C}}$ (i.e. $\bar{\mathcal{C}} \geq [\frac{1}{2}\lambda^* I_3 + \bar{K}_2]^{-1}$), and this choice does ensure

that $\bar{\tau}_0$ is indeed 'small'.

Finally, we observe that, if $\bar{K}_2$ is suitably sized, say for example, $\bar{K}_2 < \frac{1}{2}\lambda^* I_3$,

then

$$\frac{1}{[\frac{1}{2}\lambda^* I_3 + \bar{K}_2]} \leq \frac{1}{[\frac{1}{2}\lambda^* I_3]} - \frac{\bar{K}_2}{2\,[\frac{1}{2}\lambda^* I_3]^2}$$

and as $\bar{K}_2 \geq 0$ then

$$\frac{1}{[\frac{1}{2}\lambda^* I_3 + \bar{K}_2]^2} \leq \frac{1}{[\frac{1}{2}\lambda^* I_3]^2} .$$

When applied in (1.4.80), these estimates show that

$$\begin{aligned}\bar{u}(x,\bar{\tau}_0) \geq \omega^* - \frac{e^{\frac{1}{2}}\varphi^*}{[\frac{1}{2}\lambda^* I_3]\,\bar{\tau}_0} + \frac{e^{\frac{1}{2}}\bar{K}_2\varphi^*}{2\,[\frac{1}{2}\lambda^* I_3]^2\bar{\tau}_0} \\ - e\bar{E}_1\varphi^* \left\{1 + \frac{1}{[\frac{1}{2}\lambda^* I_3]^2\bar{\tau}_0^{\,2}}\right\}\end{aligned}$$

for $x \in \Omega$ (1.4.82)

In order to verify the initial condition (1.4.39) we require, in addition to (1.4.82), an estimate of the size of $u(x,\bar{\tau}_0)$. We derive this estimate by making use of our upper solution to u from Section 1.4.1.

Recalling the concluding inequality of that section, inequality (1.4.37), we have that

$$u(x,t) \le \omega^* - \frac{\mu_1 \varphi^*}{(t+t_1)} + \frac{z_1(x)}{(t+t_1)^2} + (\lambda-\lambda^*)\,[I_1 \varphi^* t + \Phi^*]$$

for $x \in \Omega,\ t$ satisfying (1.4.36) and where $(\lambda-\lambda^*) = \epsilon$ in current notation.

The condition (1.4.36) requires that t lies in a range defined as

$$\bar{t} - Z\delta^{-1} \le t \le t_1(\lambda-\lambda^*)^{-1/2} = t_1\epsilon^{-1/2}$$

where Z and t_1 are positive constants and δ is small

with $\mu_1 = (\tfrac{1}{2}\lambda^* I_3)^{-1} - \delta > 0$.

Further, t_1 has been chosen as

$$t_1 = t_0 + c(\bar{t} + t_0) = c\bar{t} + c_1 \quad \text{say,}$$

for positive constants $t_0,\ c_1$.

We require that inequality (1.4.37) holds at $t = \bar{t}_0 = \epsilon^{-1/2}\bar{\tau}_0$ and hence that

$\bar{t}_0$ lies in the range of (1.4.36), i.e. that

$$\bar{t} - Z\delta^{-1} \le \bar{\tau}_0\epsilon^{-1/2} \le t_1\epsilon^{-1/2}. \tag{1.4.83}$$

However, we see from (1.4.81) that $\bar{\tau}_0 = \bar{c}e^{\gamma}$ for some $0<\gamma<½$ and hence that the right hand inequality of (1.4.83) is automatic. The upper estimate to $u(x,t)$ given by (1.4.37) does apply at $t = \bar{t}_0$, therefore, provided

$$\bar{t} = Z\delta^{-1} \le \bar{\tau}_0 e^{-½} = \bar{c}e^{-(½-\gamma)} \ , \tag{1.4.84}$$

and

$$u(x,t_0) \le \omega^* - \frac{\mu_1\varphi^*}{(\bar{t}_0 + t_1)} + \frac{z_1(x)}{(\bar{t}_0 + t_1)^2} + \epsilon\left\{I_1\varphi^*\bar{t}_0 + \Phi^*\right\}$$

$$\text{for} \quad x\in\Omega. \tag{1.4.85}$$

On substituting for μ_1 and t_1 in (1.4.85), it follows that

$$u(x,\bar{t}_0) \le \omega^* - \frac{[½\lambda^* I_3]^{-1}\varphi^*}{[\bar{t}_0 + c\bar{t} + c_1]} + \frac{\delta\varphi^*}{[\bar{t}_0 + c\bar{t} + c_1]}$$
$$+ \frac{z_1(x)}{[\bar{t}_0 + c\bar{t} + c_1]^2} + \epsilon\left[I_1\varphi^*\bar{t}_0 + \Phi^*\right].$$

Hence as $c\bar{t}+c_1\ge 0$ and $z_1(x)\ge 0$ then

$$\begin{aligned} u(x,\bar{t}_0) &\le \omega^* - \frac{\Phi^*}{[½\lambda^* I_3]\,\bar{t}_0}\left[1-\frac{(c\bar{t}+c_1)}{\bar{t}_0}\right] \\ &\quad + \frac{\delta\varphi^*}{\bar{t}_0} + \frac{z_1(x)}{\bar{t}_0^{\,2}} + \epsilon\left[I_1\varphi^*\bar{t}_0+\Phi^*\right] \\ &= \omega^* - \frac{\varphi^*}{[½\lambda^* I_3]\,\bar{t}_0} + \frac{\delta\varphi^*}{\bar{t}_0} \\ &\quad + \frac{\left[(cZ\delta^{-1}+c_1)\varphi^* + [½\lambda^* I_3]\ z_1(x)\right]}{[½\lambda^* I_3]\,\bar{t}_0^{\,2}} + \epsilon\left[I_1\varphi^*\bar{t}_0+\Phi^*\right] \end{aligned} \tag{1.4.86}$$

on substituting for $\bar{t}$ by (1.4.84).

Finally, as $\bar{t}_0 - \epsilon^{-1/2}\bar{\tau}_0$, we are able to derive an upper estimate for $u(x,\bar{t}_0)$ whereby

$$u(x,\bar{t}_0) \leq \omega^* - \frac{\epsilon^{1/2}\varphi^*}{[\tfrac{1}{2}\lambda^* I_3]\ \bar{\tau}_0} + \frac{\epsilon^{1/2}\delta\varphi^*}{\bar{\tau}_0}$$
$$+ \frac{\epsilon\left[(cZ\delta^{-1}+c_1)\varphi^* + [\tfrac{1}{2}\lambda^* I_3]\, z_1\right]}{[\tfrac{1}{2}\lambda^* I_3]\,\bar{\tau}_0{}^2}$$
$$+ \epsilon\left[I_1\varphi^*\epsilon^{-1/2}\bar{\tau}_0 + \Phi^*\right]$$
$$\text{for} \quad x\in\Omega \qquad (1.4.87)$$

with $\delta>0$ satisfying (1.4.84).

On combining the inequalities (1.4.82) and (1.4.87) we see that the initial condition (1.4.39) will be satisfied as required if

$$\bar{u}(x,\bar{t}_0) \geq \omega^* - \frac{\epsilon^{1/2}\varphi^*}{[\tfrac{1}{2}\lambda^* I_3]\,\bar{\tau}_0} + \frac{\epsilon^{1/2}\bar{K}_2\varphi^*}{2\,[\tfrac{1}{2}\lambda^* I_3]^2\bar{\tau}_0} - \epsilon\bar{E}_1\varphi^*\left\{1 + \frac{1}{[\tfrac{1}{2}\lambda^* I_3]^2\bar{\tau}_0{}^2}\right\}$$
$$\geq \omega^* - \frac{\epsilon^{1/2}\varphi^*}{[\tfrac{1}{2}\lambda^* I_3]\,\bar{\tau}_0} + \frac{\epsilon^{1/2}\delta\varphi^*}{\bar{\tau}_0}$$
$$+ \frac{\epsilon}{[\tfrac{1}{2}{}_2\lambda^* I_3]\,\bar{\tau}_0^2}\{(cZ\delta^{-1} + c_1)\varphi^* + [\tfrac{1}{2}\lambda^* I_3]\, z_1\}$$
$$+ \epsilon\left\{I_1\varphi^*\bar{\tau}_0\epsilon^{-1/2} + \Phi^*\right\}$$
$$\geq u(x,\bar{\tau}_0) \quad \text{for} \quad x\in\Omega. \qquad (1.4.88)$$

It follows, therefore, that $\bar{K}_2$ must be chosen such that

$$\bar{K}_2\varphi^* \geq 2[\tfrac{1}{2}\lambda^* I_3]^2\left\{\delta\varphi^* + \frac{\epsilon^{1/2}}{[\tfrac{1}{2}\lambda^* I_3]\,\bar{\tau}_0}\{(cZ\delta^{-1} + c_1)\varphi^* + [\tfrac{1}{2}\lambda^* I_3]z_1\}\right.$$

$$\left. + \epsilon^{1/2}\bar{\tau}_0\{I_1\varphi^*\epsilon^{-1/2}\bar{\tau} + \Phi^*\} + \epsilon^{1/2}\bar{E}_1\varphi^*\bar{\tau}_0\left\{1 + \frac{1}{[\tfrac{1}{2}\lambda^* I_3]^2\bar{\tau}_0^2}\right\}\right\}, \qquad (1.4.89)$$

and as $\sup_{x\in\Omega}\left[\frac{z_1(x)}{\varphi^*(x)}\right]$ and $\sup_{x\in\Omega}\left[\frac{\Phi^*(x)}{\varphi^*(x)}\right]$ are both finite, then if

$$\overline{K_2} \geq B_1\delta + \frac{B_2\epsilon^{1/2}}{\overline{\tau_0}\delta} + \frac{B_3\epsilon^{1/2}}{\overline{\tau_0}} + B_4\overline{\tau_0^2} + B_5\epsilon^{1/2}\overline{\tau_0} \tag{1.4.89}$$

for suitably defined positive constants $B_1 - B_5$, then (1.4.88) will hold as required.

We require, from (1.4.82), that δ and $\overline{\tau}_0$ must satisfy the relationship

$$\epsilon^{-1/2}\overline{\tau}_0 \geq Z\delta^{-1}$$

for positive order one constant Z.

To ensure satisfaction of this inequality we choose δ such that

$$\delta^{-1} = \epsilon^{(-1/2+p)}\overline{\tau_0} \tag{1.4.91}$$

for some $p > 0$. With this δ, the requirement (1.4.90) of $\overline{K_2}$ becomes

$$\overline{K_2} \geq B_1\epsilon^{1/2-p}\overline{\tau}_0 + B_2\epsilon^{p} + B_3\frac{\epsilon^{1/2}}{\overline{\tau}_0} + B_4\overline{\tau}_0^2 + B_5\epsilon^{1/2}\overline{\tau}_0$$

which, for $\overline{\tau}_0$ as defined by (1.4.81), is automatic if

$$\overline{K_2} \geq B_1'\epsilon^{1/2-(p+\gamma)} + B_2'\epsilon^{p} + B_3'\epsilon^{1/2-\gamma} + B_4'\epsilon^{2\gamma} \tag{1.4.92}$$

for some $B_1' - B_4'$.

It follows, therefore, that $\overline{K_2}$ chosen to satisfy (1.4.92) will ensure satisfaction of the initial condition (1.4.39) as required. In addition, $\overline{K_1}$ and $\overline{K_2}$ chosen according to (1.4.79) enables us to satisfy the inequality (1.4.38) for $x \in \Omega$ and $\epsilon^{-\frac{1}{2}}\overline{\tau_0} = \overline{t_0} \le t \le \overline{t_1}$ where $\overline{t_1}$ is the first time after $\overline{t_0}$ at which inequality (1.4.75) fails. On combining the requirements (1.4.77) and (1.4.92) we see that, if $\overline{K_1}$ and $\overline{K_2}$ are chosen such that

$$\overline{K_1} \ge \epsilon^{\frac{1}{2}}\overline{A} \quad \text{and} \quad \overline{K_2} \ge \overline{B}.\ \max\{\epsilon^{\frac{1}{2}-(\gamma+p)}, \epsilon^{p}, \epsilon^{2\gamma}\} \tag{1.4.93}$$

for $p > 0,\ 0 < \gamma < \frac{1}{2}$ and (as $\overline{K_2}$ is 'small') $\gamma + p < \frac{1}{2},$ then $\overline{u}(x,t)$ will be an upper solution to u for all $t \in (\overline{t_0}, \overline{t_1})$. Further,

$\overline{t_0} = \epsilon^{-\frac{1}{2}}\overline{\tau_0} = \overline{c}\epsilon^{-\frac{1}{2}+\gamma}$ and $\overline{t_1}$ is the first time after $\overline{t_0}$ at which inequality (1.4.75) fails, i.e. at which $\overline{a}(\overline{\tau_1}) \ge \epsilon^{-\gamma}$ for $\overline{\tau_1} = \epsilon^{\frac{1}{2}}\overline{t_1}$

We have therefore arrived at an upper solution, $\overline{u}$, to u which exists and is finite until the first time after $\overline{t_0}$, say $\overline{t_1}$, at which inequality (1.4.75) fails. Unlike the analysis of Section 1.3.2, however, where an examination of the behaviour of u within time region III was required in order to derive an upper bound for t_b , this result is sufficient to allow us to estimate t_b from below.

Inequality (1.4.75) requires that

$$|\bar{a}(\tau)| \leq \epsilon^{-\gamma}$$

i.e. that $|\bar{C_1} \tan (\bar{C_2}\tau - \pi/2)| \leq \epsilon^{-\gamma}$ for $0 < \gamma < ½$ and for all $\tau \in (\bar{\tau}_0, \bar{\tau}_1)$,

where $\bar{t_0} = \epsilon^{-½}\bar{\tau}_0$, $\bar{t_1} = \epsilon^{-½}\bar{\tau}_1$. The time $\bar{\tau}_1$ is taken to be the first after $\bar{\tau}_0$ at which this inequality fails and clearly

$$\bar{\tau}_1 = \pi/\bar{C_2} - \bar{\tau}_F$$

for some small $\bar{\tau}_F$. For small $\bar{\tau}_F$ we have that

$$\begin{aligned} \bar{a}(\bar{\tau}_1) &= \bar{C_1}\tan\left(\bar{C_2}\bar{\tau}_1 - \frac{\pi}{2}\right) \\ &= \bar{C_1}\tan[\pi/2 - \bar{C_2}\bar{\tau}_F] \\ &\leq \frac{\bar{C_1}}{\bar{C_2}\bar{\tau}_F} \end{aligned}$$

so we see that a maximum value of $\bar{\tau}_1$ is given by

$$\overline{\tau_1} = \frac{\pi}{\overline{C_2}} = \overline{C}\epsilon^{\gamma} \quad \text{where} \quad \overline{C} \geq \frac{\overline{C_1}}{\overline{C_2}}. \tag{1.4.94}$$

Recalling from (1.4.71) that

$$\overline{C_2} = \left\{ \left[\tfrac{1}{2}\lambda^* I_3 + \overline{K_2}\right]\left[I_1 + \overline{K_1}\right] \right\}^{1/2}$$

where $\overline{K_1}$ and $\overline{K_2}$ satisfy (1.4.93) and are 'small' we see that

$$\begin{aligned} \overline{\tau}_1 &= \frac{\pi}{\overline{C_2}} - \overline{C}\epsilon^{\gamma} \\ &= \frac{\pi}{\left\{ \left[\tfrac{1}{2}\lambda^* I_3 + \overline{K_2}\right]\left[\, I_1 + \overline{k_1}\right] \right\}^{1/2}} - \overline{C}\epsilon^{\gamma} \\ &\geq \frac{\pi}{\left[\tfrac{1}{2}\lambda^* I_1 I_3\right]^{1/2}} - \overline{D}_1\overline{K}_1 - \overline{D}_2\overline{K}_2 - \overline{C}\epsilon^{\gamma} \end{aligned} \tag{1.4.95}$$

for suitably chosen $\overline{D_1}, \overline{D_2}$.

We conclude, therefore, that the upper solution established in Section 1.4.2 exists and is finite at $t = \overline{t_1} = \epsilon^{-1/2}\overline{\tau_1}$ where $\overline{\tau}_1$ satisfies the estimate (1.4.95). It follows by comparison that $t_b \geq \overline{t_1}$, which, on substituting for $\overline{k_1}$ and $\overline{k_2}$ the minimum values allowed by condition (1.4.93), yields that

$$\begin{aligned} t_b &\geq \overline{t}_1 = \epsilon^{-1/2}\,\overline{\tau}_1 \\ &\geq \frac{\pi\epsilon^{-1/2}}{\left[\tfrac{1}{2}\lambda^* I_1 I_3\right]^{1/2}} - \overline{D}_1\overline{A} - \overline{D}_2\overline{B}\max\left\{ \epsilon^{-(\gamma+p)}, \epsilon^{-1/2+p}, \epsilon^{-1/2+2\gamma} \right\} - \overline{C}\epsilon^{-1/2+\gamma}, \end{aligned} \tag{1.4.96}$$

where $\bar{A}$ and $\bar{B}$ are positive constants and $0 < p+\gamma < \frac{1}{2}, 0 < \gamma < \frac{1}{2}$ and $p > 0$.

We see therefore that there exists $s_2 > 0$ such that

$$t_b \geq \frac{\pi(\lambda-\lambda^*)^{-\frac{1}{2}}}{[\frac{1}{2}\lambda^* I_1 I_3]^{\frac{1}{2}}}[1 + \epsilon^{s_2}] \quad \text{for} \quad \lambda \to \lambda^{*+}. \tag{1.4.97}$$

Inequality (1.4.97) serves to establish the desired lower bound t_b and concludes the discussion of this section. This bound has of course been derived without any attempt to identify an upper solution to u in time region III.

Section 1.5 Conclusion

Putting together the estimates from Sections 1.3.3 and 1.4.2 we see that the blow-up time, t_b , satisfies

$$\frac{\pi(\lambda-\lambda^*)^{-\frac{1}{2}}}{\left[\frac{1}{2}\lambda^* I_1 I_3\right]^{\frac{1}{2}}}\left[1 + (\lambda-\lambda^*)^{s_2}\right] \le t_b \le \frac{\pi(\lambda-\lambda^*)^{-\frac{1}{2}}}{\left[\frac{1}{2}\lambda^* I_1 I_3\right]^{\frac{1}{2}}}\left[1 + (\lambda-\lambda^*)^{s_1}\right]$$

for some $s_1, s_2 > 0$ and as $\lambda \to \lambda^{*+}$.

This gives the desired result, previously obtained (equation (1.2.67)) from formal asymptotics, that if the initial condition u_0 satisfies equations (1.1.4), (1.2.21), (1.2.22) and (1.3.24), i.e. if :

$$u_0 \in C^1(\Omega), \text{ with } u_0 \ge 0 \text{ in } \Omega \text{ and } \frac{\partial u_0}{\partial n} + \beta u_0 = 0 \text{ on } \partial\Omega \tag{1.1.4}$$

$$u_0(x) < \omega^*(x) \text{ in } \Omega, \tag{1.2.21}$$

with

$$\frac{\partial u_0}{\partial n} > \frac{\partial \omega^*}{\partial n} \text{ or } u_0 < \omega^* \text{ on } \partial\Omega \tag{1.2.22}$$

and $$\nabla^2 u_0 + \lambda^* e^{u_0} \ge 0 \text{ for } x \in \Omega, \tag{1.3.24}$$

then

$$t_b \sim \frac{\pi(\lambda-\lambda^*)^{-\frac{1}{2}}}{\left[\frac{1}{2}\lambda^* I_1 I_3\right]^{\frac{1}{2}}} \quad \text{for } \lambda \text{ slightly larger than } \lambda^*.$$

In this expression $I_1 = \int_\Omega \varphi^* e^{\omega^*} dx$ and $I_3 = \int_\Omega \varphi^{*3} e^{\omega^*} dx$, are independent of the initial condition $u_0(x)$.

Chapter 2 The One-Dimensional Gradient Problem

2.1 Introduction

In this chapter, and in Chapter 3, we shall consider blow-up behaviour of solutions to the problem,

$$u_t = \nabla^2 u + u^p \pm u^\alpha |\nabla u|^\beta \qquad \text{in } \Omega,\ t > 0, \tag{2.1.1}$$

$$u(x,0) = \varphi(x) \qquad \text{in } \Omega, \tag{2.1.2}$$

$$u(x,t) = 0 \qquad \text{on } \partial\Omega,\ t > 0, \tag{2.1.3}$$

where Ω is a bounded region in $\mathbf{R}^N$ with smooth boundary $\partial\Omega$.

Immediately we observe that solutions to this problem may be expected to show different characteristics depending on the sign of the gradient term in (2.1.1). Hence it seems prudent to consider the cases of positive and negative gradient terms in (2.1.1) as entirely separate problems, even though the techniques used in their study will be virtually identical.

Interest in the blow-up behaviour of equations of the form of (2.1.1) stems, primarily, from a recent paper, Friedman & Lacey 1988, in which the case of a negative gradient term is studied in one-dimension. When the gradient term in (2.1.1) is of negative sign, it will have a damping effect and will work against blow-up. In this case it is not immediately obvious if the problem (2.1.1)-(2.1.3) will have solutions which blow up in a finite time. Further, if blow-up does occur, then it may be anticipated that the gradient term will play an influential role in determining the profile of any solutions, and hence also in governing where blow-up may take place.

Initially, the work of Friedman & Lacey considered the particular example of

$\alpha = \beta = 1$ (so that $u^{\alpha} | \nabla u |^{\beta} = uu_x$) and addressed both these points. The techniques used in this analysis were then extended to cover a number of other, more general, gradient terms, although this work restricted itself to the cases where one of the parameters α or β is equal to unity.

It is relatively straightforward to establish when, in the case of negative gradient term, the problem (2.1.1)-(2.1.3) will not, in terms of the parameters α, β and p , have solutions which exhibit finite-time blow-up (c.f. Remark 2.2.2). Hence, the work of Friedman and Lacey provides a relatively complete description of the blow-up behaviour possible for the equations considered.

In the first instance, this work sought to extend the techniques developed in Friedman & Lacey 1988 to a more general one-dimensional problem. This process can be considered relatively successful and, as it forms the basis of all subsequent work in this section, and illustrates how the techniques of Friedman & Lacey 1988 are used to establish finite-time blow-up, is included in Section 2.2.1. As already indicated by the extensions of Friedman and Lacey, however, unless $\beta = 1$, a complete description, in terms of the parameters α, β and p , of when finite-time blow-up may be anticipated, is not available by this route.

The resolution of this point consequently became the primary objective of Section 2.2.2.

Having established the existence of finite-time blow-up for any of the considered problems, we next consider the influence of the gradient term in determining the characteristics of the blow-up sets. The work of Friedman & Lacey 1988 also provides the basic techniques and arguments used here in to investigate this topic, the detail of which, for a one-dimensional problem with a negative gradient term is included in Section 2.3.

Finally, having established that finite-time blow-up can occur, and that this blow-up can be identified as occurring at a single point within the interior of the considered region, Section 2.4 is able to derive an estimate of the rate at which single point blow-up will take place.

In Sections 2.5-2.7 consider the one-dimensional form of equation (2.1.1) in which the gradient term is of positive sign. The study of this problem is primarily motivated by observations made in Section 2.3.

When in the one dimensional problem the gradient is an odd function of the gradient (ie when $- u^{\alpha} |u_x|^{\beta} = - u^{\alpha} |u_x|^{\beta-1} u_x$) the analysis of Section 2.3 finds that it can be the positive nature of this term which determines how much information is available regarding the blow-up set. This observation leads to the conclusion that interesting blow-up behaviour may be possible for the problem (2.1.1)-(2.1.3) when the gradient term is of positive sign.

When the gradient term in (2.1.1) is wholly positive, we anticipate that it will help to promote blow up. Indeed, in this case, finite-time blow-up for solutions to (2.1.1)-(2.1.3) can be established by a relatively straightforward comparison process, for a large range of initial data. However, although a positive gradient term may help promote blow-up, it will also, as with a negative gradient term, play what could be an important role in determining the shape of solutions.

Hence, where a negative gradient term will tend to focus solutions towards a maximum, a positive gradient term will have the opposite effect and may work against single point blow-up by encouraging flatter solutions.

The analysis of Sections 2.5-2.7 (in which the one-dimensional positive gradient problem is considered) is virtually identical in structure to that of Sections 2.2-2.4. In this case, however, because the existence of blow-up is relatively straightforward to establish, the significant results are those of Sections 2.6 and 2.7 in which the form of blow-up is investigated.

Section 2.2 Existence of blow-up for a negative gradient term

2.2.1 Blow-up using the techniques of Friedman and Lacey

Throughout Sections 2.2-2.4 we shall consider the problem (2.1.1)-(2.1.3) in one-dimension and where the gradient term is of negative sign. Hence, $u\,(x,t)$ denotes the solution to

$$u_t = u_{xx} + u^p - u^\alpha u_x^\beta, \quad \text{in} \quad (-a,a)\ ,\ t>0, \tag{2.2.1}$$

$$u(x,0) = \varphi(x) > 0, \quad \text{in} \quad (-a,a), \tag{2.2.2}$$

$$u(\pm a,t) = 0, \qquad t > 0. \tag{2.2.3}$$

In this case we shall also asume that α, β and p satisfy the conditions

$$p > 1\ ,\ \alpha\geq 0 \ \text{ and } \ \beta \geq 1. \tag{2.2.4}$$

In the one-dimensional problem, there are two choices for the form of the gradient term u_x^β in (2.2.1) :

either (i) $u_x^\beta = |\,u_x\,|^\beta$, or (ii) $u_x^\beta = |\,u_x\,|^{\beta-1}u_x$.

Most of the results examined in this section will be independent of which form this gradient term takes, although there are exceptions which are detailed later. In Friedman & Lacey 1988, this problem was considered for $\alpha = \beta = 1$, i.e. equation (2.2.1) took the form,

$$u_t = u_{xx} + u^p - uu_x\ ,$$

and it was shown that, if the initial condition $\varphi(x)$ satisfies

$$\varphi \in C^0[-a,a], \quad \varphi \geq 0, \quad \varphi(\pm a) = 0, \tag{2.2.5}$$

then

(a) finite-time blow up of u occurs if $p > 2$, and if φ is 'large enough' compared to a, and

(b) there is no blow up if $p \leq 2$.

The methods used to obtain these results are equally applicable to the more general one-dimensional equations (2.2.1)-(2.2.3) and are illustrated below.

If the initial condition $\varphi(x)$ satisfies condition (2.2.5), then by standard methods the solution to (2.2.1)-(2.2.3) can be shown to have a unique solution at least until some small time T_0 and that, by the maximum principle, $u > 0$, if $-a < x < a,\ 0 < t \leq T_0$. If the solution cannot be extended step-by-step to all $t > 0$, then there must exist a finite time T such that the solution exist, and is positive, for $0 < t < T$ and

$$\limsup_{t \to T} \sup_{(|x| \leq a)} u(x,t) = +\infty .$$

If this is the case we say that u blows up at time T.

We next show that the solution to equations (2.2.1)-(2.2.4) can, for appropriate initial data, exhibit finite-time blow-up, and begin by considering $v(x,t)$, the solution to,

$$v_t - v_{xx} + v^p, \qquad \text{in} \qquad (-b,b) \;,\; t>0, \tag{2.2.6}$$

$$v(x,0) - \psi(x) \qquad \text{in} \qquad (-b,b), \tag{2.2.7}$$

$$v(\pm b,t) - 0, \qquad t > 0. \tag{2.2.8}$$

If ψ is sufficiently large, then from the results of Lacey 1983, for example, $v(x,t)$ will blow up in a finite-time, say t_0. Further, if $\psi(x) - \psi(-x)$, with $\psi(\pm b) - 0$ then $v(x,t) - v(-x,t)$ and, using the maximum principle, one can establish that

$$v_x < 0 \quad \text{if} \quad 0 < x < b,\; 0 < t < t_0. \tag{2.2.9}$$

From the results of Friedman & MacLeod 1985, the estimates

$$v(x,t) \le C\,[t_0-t]^{-1/(p-1)} \;, \text{ for some } \; C > 0 \;, \tag{2.2.10}$$

and

$$\tfrac{1}{2}|\nabla v|^2 \le \int_{v(x,t)}^{v(0,t)} s^p ds \;, \tag{2.2.11}$$

where

$$v(0,t) - \max_{(|x|\le b)} v(x,t) \tag{2.2.12}$$

are also valid for v .

Thus, if $r(t)$ is defined as

$$r(t) = \int_0^t m(\tau)^\gamma d\tau, \quad r_0 = r(t_0) \tag{2.2.13}$$

where

$$m(t) = v(0,t) = \max_{(|x| \le b)} v(x,t) \quad , \tag{2.2.14}$$

then

$$r_0 < \infty \quad \text{provided} \quad \gamma < p - 1 \ . \tag{2.2.15}$$

Introduce regions

$$R_1 = \left\{ \delta(t) < x < b + \delta(t), 0 < t < t_0 \right\}$$

$$R_2 = \left\{ -\delta(t) < x < \delta(t), 0 < t < t_0 \right\}$$

$$R_3 = \left\{ -b - \delta(t) < x < -\delta(t), 0 < t < t_0 \right\}$$

and the function w such that

$$w(x,t) = v(x - \delta(t), t), \qquad \text{in } R_1,$$

$$w(x,t) = v(0,t) = m(t), \qquad \text{in } R_2,$$

$$w(x,t) = v(x + \delta(t), t), \qquad \text{in } R_3,$$

where $\delta(t) = r_0 - r(t)$, so that $\delta'(t) = -m^\gamma(t)$, and propose that, under appropriate conditions on the values of parameters α and β , and provided $\varphi(x)$ is chosen suitably, $w(x,t)$ may be a subsolution to $u(x,t)$, the solution to (2.2.1)-(2.2.4).

In R_1 ,

$$\begin{aligned} w_t - w_{xx} - w^p + w^\alpha w_x^\beta &= v_t - \delta' v_x - v_{xx} - v^p + v^\alpha v_x^\beta \\ &= -\delta' v_x + v^\alpha v_x^\beta \end{aligned} \tag{2.2.16}$$

so that, if $v_x^\beta = |v_x|^{\beta-1} v_x$, the right hand side of (2.2.16) becomes

$$v_x(v^\alpha |v_x|^{\beta-1} + m^\gamma(t)) \ , \tag{2.2.17}$$

which is less than or equal to zero, as $v_x < 0$ in R_1 from (2.2.9).

Alternatively, if $v_x^\beta = |v_x|^\beta$, then the right hand side of (2.2.16) becomes

$$v^\alpha |v_x|^\beta + m^\gamma(t) v_x = |v_x| (v^\alpha |v_x|^{\beta-1} - m^\gamma(t)) \ . \tag{2.2.17}$$

If inequality (2.2.11) is then used to substitute for $|v_x|$, and as $v(x,t) \le m(t)$, (2.2.17) will in turn be less than or equal to

$$|v_x| (Cm^{\alpha + \frac{1}{2}(p+1)(\beta-1)} - m^\gamma) \ , \tag{2.2.18}$$

for some constant c, provided $\alpha \geq 0$ and $\beta \geq 1$.

This expression will therefore be less than or equal to zero, as required, provided

$$\alpha + \tfrac{1}{2}(p+1)(\beta-1) < \gamma < p-1 , \tag{2.2.19}$$

and m is sufficiently large.

Next, in R_2,

$$w_t - w_{xx} - w^p + w^\alpha w_x^\beta = (v_t - v_{xx} - v^p + v^\alpha v_x^\beta)(0,t) \leq 0 ,$$

as $v_{xx} \leq 0$ and $v_x = 0$ at $x = 0$.

Finally, in R_3,

$$\begin{aligned} w_t - w_{xx} - w^p + w^\alpha w_x^\beta &= v_t + \delta' v_x - v_{xx} - v^p + v^\alpha v_x^\beta \\ &= v^\alpha v_x^\beta - m^\gamma(t)\, v_x . \end{aligned} \tag{2.2.20}$$

As $v_x > 0$ in R_3, the argument used in R_1 for the case $v_x^\beta = |v_x|^\beta$ applies

directly here and can be used to show that the right hand side of equation (2.2.20) is less than or equal to zero provided condition (2.2.19) holds, i.e.

$$\alpha \geq 0, \quad \beta \geq 1, \quad \alpha + \tfrac{1}{2}(p+1)(\beta-1) < p-1 ,$$

and $m(t)$ is ‘large enough’.

As $m(t) = \max_{(|x| \leq b)} v(x,t)$, and, from the maximum principle, as

$v(x,t) \geq v(x,0) = \psi$, $m(t)$ can be made as large as required by suitable choice of ψ . Observing next that both w and w_x are continuous across the boundaries $\partial R_1 \cap \partial R_2$ and $\partial R_2 \cap \partial R_3$, with $w_x = 0$ on these curves, we conclude that w satisfies the requirements necessary to be a subsolution to u in the interior, Ω , of the set $\bar{R}_1 \cup \bar{R}_2 \cup \bar{R}_3$.

Further, w is continuous on the parabolic boundary $\partial_p\Omega$ of Ω with

$$w(x,0) = \psi(x) \text{ , for } -b' < x < b', \quad (b' = b + \delta(0) = b + r_0) ,$$

$$\text{and } w = 0 \text{ elsewhere on } \partial_p\Omega .$$

If $a > b'$ and $\varphi(x) > \psi(x)$ for $-b' < x < b'$, then by comparison

$$u(x,t) \geq w(x,t) . \tag{2.2.21}$$

The results of Lacey 1983 therefore lead to the conclusion that, if ψ is large enough compared to b , and hence if $\varphi(x)$ is large enough compared to a , then $v(x,t)$ exhibits finite time blow-up and that $w(x,t)$ is a subsolution to u which blows up in a finite time.

We have therefore proved the following:-

Theorem 2.2.1

If φ is large enough compared to a (so that inequality (2.2.21) is valid for a function $w(x,t)$ which exhibits finite time blow-up) and if $\alpha + \frac{1}{2}(p+1)(\beta - 1) < p - 1$, $\alpha \geq 0$, and $\beta \geq 1$, then u , the solution to (2.2.1)-(2.2.4), blows up in a finite time less than or equal to t_0 .

Remark 2.2.2

There is no blow-up if $\alpha + \beta \geq p$.

Proof

The function

$$W(x,t) = Ae^{b(x+a)} , \tag{2.2.22}$$

can be shown, if $b > 1$, and A is large enough compared to b , to be a supersolution to u provided $\alpha + \beta \geq p > 1$.

$$W_t - W_{xx} - W^p + W^\alpha W_x^\beta = A^p e^{pb(x+a)}\{A^{(\alpha+\beta-p)} b^\beta e^{(\alpha+\beta-p)b(x+a)} - 1\} \quad -Ab^2 e^{b(x+a)} . \tag{2.2.23}$$

If then $\alpha + \beta \geq p$ and $A > 1$, the right hand side of equation (1.2.23) is greater than or equal to

$$A^p e^{pb(x+a)}(b^\beta - 1) - Ab^2 e^{b(x+a)} = Ae^{b(x+a)}\{A^{p-1}e^{(p-1)b(x+a)}(b^\beta - 1) - b^2\} ,$$

which, if $b,p > 1$ and A is large enough compared to b , is strictly greater than zero for all $x \in (-a,a)$.

Also, $w(\pm a, t) > A > 0$ and $w(x, 0)$ can be made as large as required by appropriate choice of A.

It follows that w is a supersolution to u which is bounded for all times t and hence Remark 2.2.2.

2.2.2 A 'stronger' blow-up result

Theorem 2.2.1 and Remark 2.2.2 of Section 2.2.1 illustrate the direct application of the methods of Friedman and Lacey to the problem (2.2.1)-(2.2.4). The question remains, however, as to the blow-up behaviour of solutions to (2.2.1)-(2.2.4) when α, β and p are outwith the scope of either of these results, i.e. when α, β and p are in the range

$$\frac{(3p-2\alpha-1)}{(p+1)} \leq \beta < p - \alpha, \tag{2.2.24}$$

with $p > 1, \alpha \geq 0,$ and $\beta \geq 1.$

To try and answer this question, it is intended to repeat the analysis of Friedman & Lacey 1988, illustrated in Section 2.2.1, with the modification that an alternative function is chosen to play the role of $v(x,t),$ the solution to (2.2.6) - (2.2.7).

In order to do this, such an alternative function must satisfy a number of requirements, the most important of which are:-

1. It must itself exhibit finite time blow up, and
2. Its gradient must satisfy an estimate similar to (2.2.11).

We here propose the solution to problem (2.2.1)-(2.2.3), with $\beta = 1,$ and $\alpha = q,$ say, as this alternative, which we shall henceforth call $z(x,t),$ i.e. $z(x,t)$ satisfies:-

$$z_t = z_{xx} + z^p - z^q|z_x| \quad \text{in} \quad (-b,b), t > 0, \tag{2.2.25}$$

$$z(x,0) = \psi(x) \qquad \text{in} \quad (-b,b), \tag{2.2.26}$$

$$z(\pm b,t) = 0, \qquad t > 0, \tag{2.2.27}$$

where

$$\psi \in C^0[-b,b], \quad \psi \geq 0, \quad \text{and} \quad \psi(\pm b) = 0. \tag{2.2.28}$$

The results of Theorem 2.2.1 and Remark 2.2.2 when applied to this equation yield that $z(x,t)$ will blow up in a finite time, say $T,$ provided

$$0 \leq q < p - 1, \tag{2.2.29}$$

and ψ is large enough compared to $b,$ and that $z(x,t)$ will remain bounded for all times t if

$$q \geq p - 1 > 0. \tag{2.2.30}$$

An upper bound for the gradient of the solution which also applies to the more general N-dimensional problem can be found.

This result takes the form that, if $u(x,t)$ is the solution to (2.1.1)-(2.1.3), then

$$|\nabla u|^2 \leq \{Cu^m + u^k - Au^l + B\}^2, \tag{2.2.31}$$

where C, A, B are some positive constants, with B large compared to $A,$ and provided

(i) $\beta > 2(p - \alpha)/(p + 1)$

(ii) $C^\beta[\alpha + n(\alpha + m(\beta - 1))] > p$

(iii) $C \geq 1$ and $m = (p - \alpha)/\beta > k > l > 1$,

with k chosen 'close enough' to m, and n the largest integer satisfying

$n \leq \beta$.

The proof of this result is straightforward but technical, and is included separately in Appendix A.

When applied to $z(x, t)$, however, this result reduces to

$$|z_x|^2 \leq \{Cz^m + z^k - Az^l + B\}^2, \tag{2.2.32}$$

for positive constants A, B and C with B large enough compared to A,

and

(i) $q > \frac{1}{2}(p - 1)$,

(ii) $q > \dfrac{p}{2C}$,

(iii) $C \geq 1$ and $m = p - q > k > l > 1$,

and k chosen close enough to $p - q$.

If we make the additional assumption that

$$\psi(x) \in C^1[-b, b], \quad \psi(x) = \psi(-x), \quad \text{with} \quad \psi'(x) < 0, \quad \text{for} \quad 0 < x < b, \tag{2.2.33}$$

then

$$z(x, t) = z(-x, t), \tag{2.2.34}$$

and using the maximum principle, one can establish that

$$z_x(x,t) < 0 \quad \text{if} \quad 0 < x < b, 0 < t < T, \tag{2.2.35}$$

provided $p \geq 1$, and $q \geq 0$.

Now define the function $\delta(t)$ as

$$\delta(t) = M(T - t), \quad \text{for} \quad 0 < t < T, \tag{2.2.36}$$

where M is some positive constant, regions R_1-R_3 as

$$R_1 = \{\delta(t) < x < b + \delta(t), 0 < t < T\}$$

$$R_2 = \{-\delta(t) < x < \delta(t), 0 < t < T\}$$

$$R_3 = \{-b - \delta(t) < x < -\delta(t), 0 < t < T\},$$

and the function $w(x,t)$ by

$$w(x,t) = z(x - \delta(t), t), \quad \text{in} \quad R_1,$$

$$w(x,t) = z(0,t), \quad \text{in} \quad R_2,$$

and $$w(x,t) = z(x + \delta(t), t), \quad \text{in} \quad R_3.$$

We propose that, for a large class of initial conditions $\varphi(x)$, $w(x,t)$ will be a subsolution to u.

In R_1,

$$w_t - w_{xx} - w^p + w^\alpha w_x^\beta = z_t - \delta' z_x - z_{xx} - z^p + z^\alpha z_x^\beta$$
$$= z^\alpha z_x^\beta - \delta' z_x - z^q |z_x|. \tag{2.2.37}$$

If $z_x^\beta = |z_x|^{\beta-1} z_x$, then the right hand side of (2.2.37) becomes

$$z^\alpha |z_x|^{\beta-1} z_x + M z_x - z^q |z_x|, \tag{2.2.38}$$

as $\delta' = -M$ from (2.2.36).

From (2.2.35), however, $z_x < 0$ in R_1, so that (2.2.38) is less than or equal to zero.

Alternatively, if $z_x^\beta = |z_x|^\beta$, then the right hand side of (2.2.37) becomes,

$$z^\alpha |z_x|^\beta + M z_x - z^q |z_x| = |z_x| \{ z^\alpha |z_x|^{\beta-1} - M - z^q \}. \tag{2.2.39}$$

If inequality (2.2.32) is used to estimate z_x, and if $\beta \geq 1$, then the right hand side of equation (2.2.39) is less than or equal to

$$|z_x| \{ z^\alpha (C z^m + z^k - A z^l + B)^{\beta-1} - M - z^q \}, \tag{2.2.40}$$

which is also less than or equal to zero if either

(a) z is small compared to M, or if

(b) $\alpha + m(\beta - 1) < q$, and z is large compared to C, A, and B.

As there is no relationship between the relative sizes of M, and A, B, and C, choosing M large enough ensures that one of these conditions will always be satisfied, and (2.2.40) will be less than or equal to zero, provided

$$\alpha + m(\beta - 1) < q. \tag{2.2.41}$$

Next, in R_2,

$$w_t - w_{xx} - w^p + w^\alpha w_x^\beta = \{z_t - z_{xx} - z^p + z^\alpha z_x^\beta\}(0, t) \le 0$$

as $z_{xx} \le 0$ and $z_x = 0$ at $x = 0, 0 < t < T$.

Finally, in R_3,

$$\begin{aligned} w_t - w_{xx} - w^p + w^\alpha w_x^\beta &= z_t + \delta' z_x - z_{xx} - z^p + z^\alpha z_x^\beta \\ &= \delta' z_x - z^q |z_x| + z^\alpha z_x^\beta \\ &= |z_x| \{z^\alpha z_x^{\beta-1} - z^q - M\} \end{aligned} \tag{2.2.42}$$

as $z_x > 0$ in R_3.

The right hand side of equation (2.2.42) can be shown to be less than or equal to zero, by exactly the argument used in R_1 for the case $z_x^\beta = |z_x|^\beta$, if M is large enough, $\beta \ge 1$, and

$$\alpha + m(\beta - 1) < q. \tag{2.2.43}$$

As w and w_x are once again observed to be continuous across the boundaries $\partial R_1 \cap \partial R_2$ and $\partial R_2 \cap \partial R_3$, with $w_x = 0$ on these curves, we conclude that w satisfies the necessary requirements to be a subsolution to u in the interior, Ω , of the set $\bar{R}_1 \cup \bar{R}_2 \cup \bar{R}_3$.

Further, w is also continuous on the parabolic boundary $\partial_p \Omega$, with

$$w(x,0) = \psi(x), \quad \text{for} \quad -b' < x < b'$$

and $\quad w = 0, \quad$ elsewhere on $\partial_p \Omega$,

where $b' = b + \delta(0) = b + MT$.

Hence, if $a > b'$ and $\varphi(x) > \psi(x)$ for $-b' < x < b'$ then by comparison

$$u(x,t) \geq w(x,t), \tag{2.2.44}$$

provided,

(i) $\quad \beta \geq 1, \qquad$ (2.2.41)

(ii) $\quad \alpha + m(\beta - 1) < q, \qquad$ (2.2.38)

and to allow the use of estimate (2.2.32)

(iii) $\quad q > \frac{1}{2}(p - 1)$

(iv) $\quad q > p/2C$

(v) $C \geq 1$ and $m - p - q > k > l > 1$.

In addition, from Theorem 2.2.1, $z(x,t)$ and hence $w(x,t)$ will blow up in finite time provided $\psi(x)$ is large enough compared to b and $p > q + 1$.

If q is therefore chosen close enough to $p-1$ the above conditions reduce to

(i) $\beta \geq 1$,

(ii) $\alpha + \beta < p$,

(iii) $p > 1$,

(iv) $p > \frac{2C}{2C-1}$,

(v) $C \geq 1$ and $p > 1$,

which can be satisfied, for all $p > 1$, provided C is chosen appropriately large.

We may therefore conclude

Theorem 2.2.3

If φ is large enough compared to a, (so that $w(x,t)$ is a subsolution which blows up) the solution to (2.2.1)-(2.2.4) will blow up in a finite time provided,

$$\alpha \geq 0, \quad \beta \geq 1, \quad p > 1, \quad \text{and} \quad \alpha + \beta < p. \tag{2.2.45}$$

We now investigate the nature of this blow-up.

Section 2.3 Identification of the blow-up sets

In this section we consider a solution (2.2.1)-(2.2.4) which satisfies the requirements of Theorem 2.2.1 and which does therefore exhibit finite time blow up, at a time T say.

To recap, this function $u(x,t)$ satisfies

$$u_t = u_{xx} + u^p - u^\alpha u_x^\beta \qquad \text{in} \quad (-a,a),\, t > 0, \tag{2.2.1}$$

$$u(x,0) = \varphi(x), \qquad \text{in} \quad (-a,a), \tag{2.2.2}$$

$$u(\pm a,t) = 0, \qquad \text{for} \quad t > 0, \tag{2.2.3}$$

where the term u_x^β is chosen as either $|u_x|^\beta$ or $|u_x|^{\beta-1}u_x$, α,β and p are such that

$$\alpha \geq 0,\ \beta \geq 1,\ p > 1 \ \text{and} \ p > \alpha + \beta, \tag{2.2.45}$$

the initial value $\varphi(x)$ satisfies

$$\varphi \in C^0[-a,a],\ \ \varphi \geq 0 \ \text{and} \ \varphi(\pm a) = 0, \tag{2.2.5}$$

and is assumed 'large enough' compared to a (from Theorem 2.2.3) .

To continue, we henceforth also assume that $\varphi(x)$ satisfies

$$\varphi \in C^1[-a,a], \tag{2.3.1}$$

with $\varphi'(x) \geq 0$ for $-a \leq x \leq x_0$, $\varphi'(x) \leq 0$ for $x_0 \leq x \leq a$, (2.3.2) for some $x_0 \in (-a, a)$, and

$$\varphi(x) \geq \varphi(-x) \quad \text{for} \quad x \geq 0. \tag{2.3.3}$$

Under these assumptions we are able to show that such a solution will, for a large range of values of the parameters α and β, blow up at a single point.

2.3.1 No blow-up for $x < 0$

We begin by introducing a number of Lemmas.

Lemma 2.3.1

There exists a continuous function $s(t)$ with $-a < s(t) < a$ for $0 \le t \le T$

such that

$$u_x(x,t) > 0 \quad \text{if } -a < x < s(t), 0 < t < T$$

and

$$u_x(x,t) < 0 \quad \text{if } s(t) < x < a, 0 < t < T. \tag{2.3.4}$$

Proof

This result can be established by the argument used in Friedman & MacLeod 1985 for equations of the form of (2.2.6) although an alternative proof is described below.

As $u(\pm a,t) = 0$, there exists at least one function $s(t)$ with

$$u_x(s(t),t) = 0 \quad \text{for } 0 < t < T, \quad \text{and } s(0) = x_0. \tag{2.3.5}$$

We define the regions I_1 and I_2 as

$$I_1 = \{s(t) < x < a, 0 < t < T\} \quad \text{and} \quad I_2 = \{-a < x < s(t), 0 < t < T\}.$$

On differentiating (2.2.1) by x we see that, if $w(x,t) = u_x(x,t)$, then w

satisfies

$$w_t = w_{xx} + pu^{p-1}w - \alpha u^{\alpha-1}w^{\beta+1} - \beta u^{\alpha}w^{\beta-1}w_x \quad \text{in } (-a,a), 0 < t < T, \tag{2.3.6}$$

with $w(-a,t) > 0,\ w(s(t),t) = 0,\ w(a,t) < 0$ for $0 < t < T$ (2.3.7)

and $w(x,0) = \varphi'(x)$ satisfying (2.3.2). (2.3.8)

Further, as $u \neq 0$ throughout the interior of the region $(-a,a)x(0,T)$, and as $\beta \geq 1$ by (2.2.4) it follows that (2.3.6) may be written as

$$w_t = w_{xx} + c_1 w + c_2 w_x \tag{2.3.9}$$

for c_1 and c_2 bounded within $(-a,a)x(0,T)$.

The maximum principle may therefore be applied to (2.3.9) within I_1 to allow the conclusion that a positive maximum of w is impossible within the interior of this set. The conditions (2.3.7) and (2.3.8) also establish that $w \leq 0$ on the parabolic boundary of I_1 and hence that

$$w = u_x < 0 \quad \text{within} \quad I_1.$$

Similarly it can be shown that

$$w = u_x > 0 \quad \text{within} \quad I_2,$$

and hence Lemma 2.3.1

Lemma 2.3.2

If $u(x,t)$ is the solution to (2.2.1) - (2.2.4) and if $\varphi(x)$ satisfies (2.3.1) - (2.3.3) then

$$u(x,t) \geq u(-x,t) \quad \text{for} \quad x \geq 0 \tag{2.3.10}$$

for all α, β and p satisfying (2.2.4).

Proof

We consider the function

$$w(x,t) = u(x,t) - v(x,t) \quad \text{in} \quad R_\epsilon \tag{2.3.11}$$

where $v(x,t) = u(-x,t)$ (2.3.12)

and $R_\epsilon = \{0 < x < a, 0 < t < T - \epsilon\}$ (2.3.13)

for all $\epsilon > 0$.

From (2.3.3) it is clear that

$$w(x,0) = \varphi(x) - \varphi(-x) \geq 0 \quad \text{for} \quad x \in (0,a), \tag{2.3.14}$$

and $w(0,t) = w(a,t) = 0,$ (2.3.15)

so that $w \geq 0$ on the parabolic boundary of R_ϵ .

On differentiating (2.3.11) with respect to t we see that w satisfies

$$w_t = w_{xx} + u^p - v^p - u^\alpha u_x^\beta + v^\alpha(-v_x)^\beta \quad \text{in} \quad R_\epsilon. \tag{2.3.16}$$

If u_x^β is chosen as $|u_x|^\beta$, then equation (2.3.16) can be written as

$$w_t = w_{xx} + (u^p - v^p) - (u^\alpha - v^\alpha)\,|v_x|^\beta - v^\alpha(\,|u_x|^\beta - |v_x|^\beta\,)$$

i.e. $$w_t - w_{xx} - cw + ev^{\alpha}w_x = -(u^{\alpha} - v^{\alpha})\,|u_x|^{\beta} \qquad (2.3.17)$$

where $c = \dfrac{u^p - v^p}{u - v}$ and $e = \dfrac{|u_x|^{\beta} - |v_x|^{\beta}}{(u_x - v_x)}$ are bounded functions throughout

R_{ϵ} in light of (2.2.4).

In addition, if w takes a negative minimum at an interior point of R_{ϵ}, then

clearly

$$w = u - v < 0$$

and hence $u < v$ at such a point.

It follows from (2.3.17) that at a negative interior minimum of w, then

$$w_t - w_{xx} - cw + ev^{\alpha}w_x = -(u^{\alpha} - v^{\alpha})\,|u_x|^{\beta} > 0,$$

a contradiction of the maximum principle.

Hence a negative minimum of w is only possible on the parabolic boundary of

R_{ϵ}, although (2.3.10) and (2.3.15) also ensure that this is not the case. It

follows, therefore, that

$$w \geq 0 \text{ in } R_{\epsilon},$$

if $u_x^{\beta} = |u_x|^{\beta}$.

Alternatively, if $u_x^\beta = |u_x|^{\beta-1}u_x$, then (2.3.16) becomes

$$w_t = w_{xx} + u^p - v^p - u^\alpha |u_x|^{\beta-1}u_x - v^\alpha |v_x|^{\beta-1}v_x,$$

i.e. $$w_t = w_{xx} + cw - dv^\alpha w_x - (u^\alpha + v^\alpha)|u_x|^{\beta-1}u_x. \tag{2.3.18}$$

In this case,

$$c = \frac{u^p - v^p}{u - v} \quad \text{and} \quad d = \frac{|u_x|^{\beta-1}u_x - |v_x|^{\beta-1}v_x}{u_x - v_x}$$

are again bounded throughout R_ϵ.

We claim that $w \geq 0$ within R_ϵ is this case also.

Otherwise w takes a negative minimum at some point $(\bar{x},\bar{t})$ in R_ϵ, i.e.

$0 < \bar{x} < a,\ 0 < \bar{t} < T - \epsilon,$ and

$$u(\bar{x},\bar{t}) < v(\bar{x},\bar{t}), \quad \text{and} \quad u_x(\bar{x},\bar{t}) = v_x(\bar{x},\bar{t}). \tag{2.3.19}$$

We first consider $\bar{x} > s(\bar{t})$.

If $\bar{x} > s(\bar{t})$, then there exists a neighbourhood W of $(\bar{x},\bar{t})$ such that

$u_x < 0$ in W.

It follows from (2.3.18), therefore, that

$$w_t - w_{xx} - cw + dv^\alpha w_x = -(u^\alpha + v^\alpha)|u_x|^{\beta-1}u_x > 0 \quad \text{in } W \tag{2.3.20}$$

a contradiction to the maximum principle.

Next, as $w(x,t) > 0$ on $\bar{x} = s(\bar{t})$, we must have that $\bar{x} \neq s(\bar{t})$.

Finally, if $0 < \bar{x} < s(\bar{t})$, then from Lemma 2.3.1,

$$u_x(\bar{x},\bar{t}) > 0, \quad \text{and} \quad v_x(\bar{x},\bar{t}) = -u_x(-\bar{x},\bar{t}) < 0,$$

as $-\bar{x} < 0$, a contradiction to (2.3.19).

We conclude, that for either choice of $(u_x)^\beta$, $w \geq 0$ in R_ϵ, and hence

Lemma 2.3.2.

From Lemmas 2.3.1 and 2.3.2 we obtain

Corollary 2.3.3

$$s(t) \geq 0, \quad \text{and} \quad u_x(x,t) > 0 \quad \text{for} \quad -a < x < 0, 0 < t < T.$$

Let

$$s_- = \liminf_{t \to T} s(t), \quad \text{and} \quad s_+ = \limsup_{t \to T} s(t), \tag{2.3.21}$$

then, by Corollary 2.3.3, $s_- \geq 0$.

Theorem 2.3.4

For any $\epsilon_0 > 0$, there exists a constant $C > 0$ such that

$$u(x,t) \leq C \quad \text{if} \quad -a \leq x \leq s_- - \epsilon_0, \; 0 < t < T, \tag{2.3.22}$$

and there is no blow-up for $x < 0$, provided,

$$p > \alpha + \beta, \quad \text{and} \quad \alpha + \beta \geq 1. \tag{2.3.23}$$

Proof

Let $\delta = s_- - \frac{1}{2}\epsilon_0$, and

$$R_0 = \{ -a \leq x \leq \delta, T_0 \leq t < T \}, \tag{2.3.24}$$

where T_0 is close enough to T so that

$$s(t) > \delta \quad \text{for} \quad T_0 \leq t < T. \tag{2.3.25}$$

From Corollary 2.3.3, $u_x(x,t) > 0$ in R_0.

Now consider the function

$$J(x,t) = u_x(x,t) - \epsilon c(x) g(u(x,t)), \quad \text{in} \quad R_0, \tag{2.3.26}$$

where ϵ is any small positive number, and $c(x)$, $g(u)$ are some positive functions to be determined, with

$$c(\delta) = g(0) = 0. \tag{2.3.27}$$

We wish to show, for appropriate $c(x), g(u)$ and ϵ small enough, that

$J > 0$ in R_0.

$$\begin{aligned} J(-a,t) &= u_x(-a,t) - \epsilon c(-a) g(u(-a,t)) \\ &= u_x(-a,t) \\ &> 0 \qquad \text{(for } t \geq T_0 \text{), by (2.3.27), and Lemma 2.3.1,} \end{aligned}$$

$$J(\delta,t) = u_x(\delta,t) - \epsilon c(\delta) g(u(\delta,t))$$
$$= u_x(\delta,t)$$
$$> 0, \qquad \text{by (2.3.25), (2.3.27), and Lemma 2.3.1,}$$

and

$$J(x,T_0) = u_x(x,T_0) - \epsilon c(x) g(u(x,T_0))$$
$$> 0 \qquad \text{by (2.3.25) and Lemma 2.3.1 if } \epsilon \text{ is small enough.}$$

We therefore see that $J > 0$ on the parabolic boundary of R_0.

Next, on differentiating (2.3.26),

$$J_t = u_{xt} - \epsilon c g' u_t, \tag{2.3.28}$$

$$J_x = u_{xx} - \epsilon c' g - \epsilon c g' u_x, \tag{2.3.29}$$

and $$J_{xx} = u_{xxx} - \epsilon c'' g - 2\epsilon c' g' u_x - \epsilon c g'' u_x^2 - \epsilon c g' u_{xx}, \tag{2.3.30}$$

so that

$$\begin{aligned} J_t - J_{xx} = {} & p u^{p-1} u_x - \alpha u^{\alpha-1} u_x^{\beta+1} - \beta u^{\alpha} u_x^{\beta-1} u_{xx} \\ & - \epsilon g' c\{u^p - u^{\alpha} u_x^{\beta}\} + \epsilon c'' g(u) \\ & + 2\epsilon c' g'(u) u_x + \epsilon c g''(u) u_x^2. \end{aligned} \tag{2.3.31}$$

Using equation (2.3.29) to substitute for u_{xx}, the right hand side of (2.3.31)

becomes

$$pu^{p-1}u_x - \alpha u^{\alpha-1}u_x^{\beta+1} - \beta u^{\alpha}u_x^{\beta-1}J_x$$
$$- \epsilon\beta u^{\alpha}u_x^{\beta-1}c'g(u) - \epsilon(\beta-1)u^{\alpha}u_x^{\beta}cg'(u) - \epsilon cg'(u)u^p$$
$$+ \epsilon c''g(u) + 2\epsilon c'g'(u)u_x + \epsilon cg''(u)u_x^2.$$

(2.3.32)

If g is chosen such that $g''(u) \geq 0$, then $\epsilon cg''(u)u_x^2 \geq 0.$ Using equation (2.3.26) to substitute for u_x in (2.3.32) then yields

$$J_t - J_{xx} - CJ - DJ_x \geq S \tag{2.3.33}$$

for some functions C and D , and where

$$S = \epsilon c\{pu^{p-1}g - g'u^p\} - \epsilon^{\beta+1}c^{\beta+1}\alpha u^{\alpha-1}g^{\beta+1}$$
$$- \epsilon^{\beta}c^{\beta-1}\beta c'u^{\alpha}g^{\beta} - \epsilon^{\beta+1}c^{\beta+1}(\beta-1)u^{\alpha}g^{\beta}g'$$
$$+ \epsilon c''g + 2\epsilon^2 cc'gg'.$$

(2.3.34)

We wish to apply the maximum principle to inequality (2.3.33) in order to demonstrate that J cannot have a negative interior minimum within R_0. To do this, we must first ensure that the functions C and D will be bounded at a negative interior minimum. The terms C and D are functions of u, u_x, c, c'

and J. We assume that c and c' both remain bounded throughout R_0.

In addition, we have from Lemma 2.3.1, $u > 0$ and $u_x > 0$ within the interior

of R_0. Hence, C and D must be bounded as functions of c, c', u and

u_x throughout the interior of R_0 as required.

We next show that C and D must also be bounded at any interior minimum of

J as follows:-

Inequality (2.3.33) may be written as

$$J_t - J_{xx} - d(u,u_x)J_x \geq \sum_{i-0}^{2} c_i(u,u_x)[(J + \epsilon cg)^{\beta-1+i} - (\epsilon cg)^{\beta-1+i}]$$
$$+ c_3(u,u_x)J + S, \tag{2.3.35}$$

where $c_i,\ i - 0,1,2,3$ and $d(-D)$ are bounded within R_0.

If we assume for the moment that S, as defined by (2.3.34), can be shown to be

no less than zero through R_0, then (2.3.35) becomes

$$J_t - J_{xx} - dJ_x \geq \sum_{0}^{2} c_i [(J+\epsilon cg)^{\beta-1+i} - (\epsilon cg)^{\beta-1+i}] + c_3J. \tag{2.3.36}$$

Hence, if J attains a minimum within the interior of R_0 at which $J - 0$,

then the right hand side of (2.3.36) vanishes at such a point and this leads to a

contradiction of the maximum principle.

If follows, therefore, that J must be non-zero at any interior minimum.

Hence, if the functions c_i' are defined as

$$c_i' = \frac{c_i\left[(J+\epsilon cg)^{\beta-1+i} - (\epsilon cg)^{\beta-1+i}\right]}{J} \qquad \text{for} \quad i = 0,1,2 \ ,$$

then each c_i is necessarily bounded at an interior minimum of J and (2.3.36) becomes

$$J_t - J_{xx} - DJ_x - CJ \geq 0 \tag{2.3.37}$$

where $D = d$ and $C = \left[c_0' + c_1' + c_2' + c_3\right]$.

The maximum principle applied to (2.3.37) consequently leads to the conclusion that a non-positive minimum of the function J is impossible within R_0 .

Hence, as J is strictly greater than zero as the parabolic boundary of R_0 we see that $J > 0$ throughout R_0 .

It follows that if S as defined by equation (2.3.34) can be shown to be no less than zero within R_0 , we may conclude that J is strictly greater than zero throughout R_0 .

We proceed to choose functions $c(x)$ and $g(u)$ which ensure that s is greater than or equal to zero as required.

If we choose

$$g(u) = u^k \quad \text{and} \quad c(x) = (x-\delta)^n \tag{2.3.38}$$

with $k \geq 1$ and n positive and even, then this choice satisfies condition (2.3.37). Further, $g''(u) \geq 0$ and $c(x), c'(x)$ are bounded throughout R_0 as required. Equation (2.3.34) then yields

$$\begin{aligned} S/\epsilon c = {} & (p-k)\,u^{p+k-1} - \epsilon^{\beta} \mid x-\delta \mid^{n\beta} \alpha u^{\alpha-1+k(\beta+1)} \\ & + n\beta\epsilon^{\beta-1} \mid x-\delta \mid^{n(\beta-1)-1} u^{\alpha+k\beta} - \epsilon^{\beta} \mid x-\delta \mid^{n\beta} (\beta-1)\,k u^{\alpha-1+k(\beta+1)} \\ & + \frac{n(n-1)\,u^k}{\mid x-\delta \mid^2} - 2\epsilon n \mid x-\delta \mid^{n-1} k u^{2k-1}. \end{aligned} \tag{2.3.39}$$

It is clear from (2.3.39) therefore, that s will be greater than or equal to zero if

$$\begin{aligned} (p-k)\,u^{p+k-1} + \frac{n(n-1)\,u^k}{\mid x-\delta \mid^2} \geq {} & (\alpha + k(\beta-1))\,\epsilon^{\beta} \mid x-\delta \mid^{n\beta} u^{\alpha-1+k(\beta+1)} \\ & + 2\epsilon nk \mid x-\delta \mid^{n-1} u^{2k-1}. \end{aligned} \tag{2.3.40}$$

This inequality will clearly be satisfied if

$$\begin{aligned} (p-k)\,u^{p+k-1} \geq {} & (\alpha+k(\beta-1))\,\epsilon^{\beta} \mid x-\delta \mid^{n\beta} u^{\alpha-1+k(\beta+1)} \\ & + 2\epsilon nk \mid x-\delta \mid^{n-1} u^{2k-1}, \end{aligned}$$

which we call condition A, or if

$$\begin{aligned} \frac{n(n-1)\,u^k}{\mid x-\delta \mid^2} \geq {} & (\alpha+k(\beta-1))\,\epsilon^{\beta} \mid x-\delta \mid^{n\beta} u^{\alpha-1+k(\beta+1)} \\ & + 2\epsilon nk \mid x-\delta \mid^{n-1} u^{2k-1} \end{aligned}$$

which we call condition B.

Condition A will be satisfied if

$$p + k - 1 > (\beta+1)k + \alpha - 1 \text{ , and } p + k - 1 > 2k - 1 \text{ ,}$$

with

$$\tfrac{1}{2}(p - k)\,u^{p+k-1-((\beta+1)k+\alpha-1))} > (\alpha + k(\beta-1))\epsilon^{\beta}\,|x-\delta|^{n\beta} \text{ ,}$$

and

$$\tfrac{1}{2}(p - k)\,u^{p+k-1-(2k-1)} > 2\epsilon nk\,|x-\delta|^{n-1} \text{ ,}$$

i.e. if

$$(p - \alpha)/\beta > k \text{ , and } p > k \text{ ,}$$

with

$$u^{p-(\alpha+k\beta)} > 2(\alpha + k(\beta - 1))\epsilon^{\beta}\,|x-\delta|^{n\beta}/(p - k) \text{ ,}$$

and

$$u^{p-k} > 4\epsilon n\,|x-\delta|^{n-1}k/(p - k) \text{ ,}$$

which we call condition A' .

Condition B is satisfied if

$$k < \alpha - 1 + k(\beta+1) \text{ , and } k < 2k - 1 \text{ ,}$$

with

$$\tfrac{1}{2}n(n-1)\,|x-\delta|^{-2} > \epsilon^{\beta}\,|x-\delta|^{n\beta}u^{\alpha-1+k(\beta+1)-k} \text{ ,}$$

and

$$\tfrac{1}{2}n(n-1)\,|x-\delta|^{-2} > 2\epsilon nk\,|x-\delta|^{n-1}u^{2k-1-k} \text{ ,}$$

i.e. if

$$\alpha + k\beta > 1 \text{ and } k > 1 \text{ ,}$$

with

$$u^{\alpha+k\beta-1} < n(n-1) \,/\, \{ 2\epsilon^{\beta} \,|\, x-\delta \,|^{n\beta+2} \} \ ,$$

and

$$u^{k-1} < (n-1) \,/\, \{ 4\epsilon \,|\, x-\delta \,|^{n+1} k \} \ ,$$

which we call condition B' .

We conclude that if k is chosen to satisfy

$$(p-\alpha)/\beta > k > 1, \quad p > k, \quad \alpha + k\beta > 1 \ , \tag{2.3.41}$$

with n large and ϵ small, then at least one of A' , or B' will be true, and

S will be greater than or equal to zero for all (x, t) in R_0 .

Now, to complete the proof of Theorem 2.3.4, we have that J , as defined by

(2.3.26), satisfies

$$J > 0 \quad \text{in} \quad R_0 \ ,$$

so that

$$u_x > \epsilon(x-\delta)^n g(u) \quad \text{in} \quad R_0 \ , \tag{2.3.42}$$

where

$$\delta = s_- - \tfrac{1}{2}\epsilon_0 \ , \text{and} \quad R_0 = \{ -a \le x \le \delta, \ T_0 \le t \le T \} \ .$$

If inequality (2.3.42) is integrated from x_0 to δ , for any $-a < x_0 < \delta$, as

$u > 0$ in R_0 and $g(u) = u^k$ with $k > 1$, we obtain

$$u(x_0,t) \le \left\{ (k-1)\int_{x_0}^{\delta} \epsilon (x-\delta)^{n} dx \right\}^{-1/(k-1)}$$

$$= \{ \epsilon(k-1)\, | x_0 - \delta |^{n+1}/(n+1) \}^{-1/(k-1)} . \tag{2.3.43}$$

If δ_1 is defined as

$$\delta_1 = \delta_0 - \tfrac{1}{2}\epsilon_0 = s_- - \epsilon_0 , \tag{2.3.44}$$

and R_1 as

$$R_1 = \{ -a \le x \le \delta_1 ,\ T_0 \le t < T \} \tag{2.3.45}$$

then from (2.3.43), at any point x_1 in R_1 , we see that

$$u(x_1,t) \le \{ \epsilon(k-1)\, | \delta_1 - \delta |^{n+1}/(n+1) \}^{-1/(k-1)} = C$$

and hence Theorem 2.3.4.

We conclude that Theorem 2.3.4 holds provided there exists $k > 1$ such that

$(p - \alpha)/\beta > k$, and $\alpha + k\beta > 1$,

i.e. provided

$p > \alpha + \beta$, and $\alpha + \beta \ge 1$.

Further, if Theorem 2.3.4 holds, there is clearly no blow-up for $x < 0$.

2.3.2 Single point blow-up

We next extend Theorem 2.3.4 to the right of $x = s(t)$.

Lemma 2.3.5

In either of the cases (a)-(c), where

(a) $u_x^\beta = |u_x|^\beta$ with $p > \alpha + \beta$, and $\beta \geq 1$,

(b) $u_x^\beta = |u_x|^{\beta-1}u_x$ with $p > \alpha + \beta$ and $\beta > 1$, or

(c) $u_x^\beta = u_x$ with $p > 2\alpha + 1$,

there exists a constant $C > 0$ such that

$$u(x,t) \leq C \quad \text{if} \quad s_+ + \epsilon_0 \leq x \leq a, \quad 0 < t < T . \tag{2.3.46}$$

Proof

We proceed, as in the proof of Theorem 2.3.4, by considering the function

$$J(x,t) = u_x(x,t) + \epsilon d(x) h(u(x,t)) , \tag{2.3.47}$$

in

$$R_1 = \{ \gamma \leq x \leq a, \quad T_0 < t < T \}. \tag{2.3.48}$$

In this case, T_0 is close enough to T to ensure that

$$s(t) < \gamma \text{ , for } T_0 \leq t \leq T , \tag{2.3.49}$$

$\gamma = s_+ + \tfrac{1}{2}\epsilon_0$, ϵ is again a small number, and $d(x)$, $h(u)$ are to be determined with

$$d(\gamma) = h(0) = 0 \ . \tag{2.3.50}$$

We wish to show that, if ϵ is small enough, then $J < 0$ in R_1 .

$$\begin{aligned} J(a,t) &= u_x(a,t) + \epsilon d(a)h(u(a,t)) \\ &= u_x(a,t) \\ &< 0, \end{aligned}$$

by (2.3.49), and Lemma 2.3.1,

$$\begin{aligned} J(\gamma,t) &= u_x(\gamma,t) + \epsilon d(\gamma)h(u(\gamma,t)) \\ &= u_x(\gamma,t) \\ &< 0, \end{aligned}$$

by (2.3.49), (2.3.50) and Lemma 2.3.1,

and

$$\begin{aligned} J(x,T_0) &= u_x(x,T_0) + \epsilon d(x)h(u(x,T_0)) \\ &< 0, \end{aligned}$$

by (2.3.49), and Lemma 2.3.1 if ϵ is small enough.

Hence $J < 0$ on the parabolic boundary of R_1 .

Next, on differentiating (2.3.46), we see that

$$J_t = u_{xt} + \epsilon d h' u_t \ , \tag{2.3.51}$$

$$J_x = u_{xx} + \epsilon d' h + \epsilon d h' u_x \ , \tag{2.3.52}$$

$$J_{xx} = u_{xxx} + \epsilon d'' h + 2\epsilon d' h' u_x + \epsilon d h'' u_x^2 + \epsilon d h' u_{xx} \ , \tag{2.3.53}$$

and hence

$$J_t - J_{xx} = pu^{p-1}u_x - \alpha u^{\alpha-1}u_x^{\beta+1} - \beta u^\alpha u_x^{\beta-1}u_{xx}$$
$$+ \epsilon dh'\{u^p - u^\alpha u_x^\beta\} - \epsilon d''h - 2\epsilon d'h'u_x - \epsilon dh''u_x^2.$$
(2.3.54)

Using (2.3.52) to substitute for u_{xx} , the right hand side of equation (2.3.54) becomes

$$pu^{p-1}u_x - \alpha u^{\alpha-1}u_x^{\beta+1} - \beta u^\alpha u_x^{\beta-1}J_x$$
$$+ \epsilon\beta u^\alpha u_x^{\beta-1}d'h + \epsilon(\beta-1)u^\alpha u_x^\beta dh' + \epsilon dh'u^p$$
$$- \epsilon d''h - 2\epsilon d'h'u_x - \epsilon dh''u_x^2.$$
(2.3.55)

We assume that $\epsilon dh''u_x^2 \geq 0$, so that, on substituting for u_x by (2.3.47) in (2.3.55), we see that

$$J_t - J_{xx} - CJ - DJ_x \leq S' \qquad (2.3.56)$$

where

$$S' = -\epsilon d\{pu^{p-1}h - h'u^p\} - \epsilon^{\beta+1}d^{\beta+1}\alpha u^{\alpha-1}(-h)^{\beta+1}$$
$$+ \epsilon^\beta d^{\beta-1}d'\beta u^\alpha(-h)^{\beta-1}h + \epsilon^{\beta+1}d^{\beta+1}(\beta-1)u^\alpha(-h)^\beta h'$$
$$- \epsilon d''h + 2\epsilon^2 dd'hh'.$$
(2.3.57)

Further, as $u > 0$ and $u_x < 0$ (by Lemma 2.3.1) throughout the region R_1 , C and D are bounded as functions of u and u_x within this region. In addition, a technique similar to that used in the development of Theorem 2.3.4 may be applied to establish that if S' , as defined by (2.3.57), can be shown to be no greater than zero throughout R_1 , then from (2.3.56)

$$J_t - J_{xx} - CJ - DJ_x \leq 0 \quad \text{within} \quad R_1 \ , \tag{2.3.58}$$

for C and D bounded (as functions of J) at any interior maximum of J .

The maximum principle applied to (2.3.58) consequently rules out the possibility of a positive interior maximum of J , and further, as J is strictly less than zero on the parabolic boundary of R_1 , that $J < 0$ throughout this set.

It remains, therefore, to establish the necessary condition, that S' , as defined by (2.3.57), is no greater than zero throughout R_1 .

It is clear, however, from (2.3.57) that the two possible choices for the form of the term u_x^β must be considered separately.

We first assume that $u_x^\beta = | u_x |^{\beta-1} u_x$, in which case (2.3.57) becomes

$$\begin{aligned} S' = -\epsilon d \left[p u^{p-1} h - h' u^{p} \right] - \epsilon^{\beta+1} d^{\beta+1} \alpha u^{\alpha-1} h^{\beta+1} \\ + \epsilon^{\beta} d^{\beta-1} d' \beta u^{\alpha} h^{\beta} - \epsilon^{\beta+1} d^{\beta+1} (\beta-1) u^{\alpha} h^{\beta} h' - \epsilon d'' h + 2\epsilon^2 d d' h h'. \end{aligned} \tag{2.3.58}$$

If we choose

$$h(u) = u^{k} \ , \text{ and } \quad d(x) = (x-\gamma)^{n} \ , \tag{2.3.59}$$

for some positive constants k and n, then this choice satisfies condition (2.3.50) and $\epsilon d h'' u_x^2 \geq 0$, as required, for $k \geq 1$. Equation (2.3.58) can therefore be written as

$$\begin{aligned} S'/\epsilon d = & -(p-k)\, u^{p+k-1} + n\beta\epsilon^{\beta-1}(x-\gamma)^{n(\beta-1)-1} u^{\alpha+k\beta} \\ & - n(n-1)\, u^{k}(x-\gamma)^{-2} + 2n\epsilon(x-\gamma)^{n-1} k u^{2k-1} \\ & - \epsilon^{\beta}(x-\gamma)^{n\beta}(\alpha+k(\beta-1))\, u^{\alpha-1+k(\beta+1)}. \end{aligned} \tag{2.3.60}$$

Hence, S' will be less than or equal to zero if

$$\begin{aligned} & (p-k)\, u^{p+k-1} + n(n-1)\, u^{k}(x-\gamma)^{-2} \geq \\ & \qquad n\beta\epsilon^{\beta-1}(x-\gamma)^{n(\beta-1)-1} u^{\alpha+k\beta} + 2\epsilon n k (x-\gamma)^{n-1} u^{2k-1}. \end{aligned} \tag{2.3.61}$$

We first consider $\beta > 1$ so that, if ϵ is small, then the term $\epsilon^{\beta-1}$ is small also. In this case, inequality (2.3.61) will be satisfied if either

$$(p-k)\, u^{p+k-1} \geq n\beta\epsilon^{\beta-1}(x-\gamma)^{n(\beta-1)-1} u^{\alpha+k\beta} + 2\epsilon n k(x-\gamma)^{n-1} u^{2k-1} ,$$

which we call condition C, or if

$$n(n-1)\, u^{k}(x-\gamma)^{-2} \geq n\beta\epsilon^{\beta-1}(x-\gamma)^{n(\beta-1)-1} u^{\alpha+k\beta} + 2\epsilon n k(x-\gamma)^{n-1} u^{2k-1}$$

which we call condition D.

Condition C is satisfied provided,

$$p+k-1 > \alpha+k\beta \ , \text{ and } \ p+k-1 > 2k-1 \ ,$$

with

$$\tfrac{1}{2}(p-k)\,u^{p+k-1} \geq n\beta\epsilon^{\beta-1}(x-\gamma)^{n(\beta-1)-1}u^{\alpha+k\beta} \quad ,$$

and

$$\tfrac{1}{2}(p-k)\,u^{p+k-1} \geq 2\epsilon nk(x-\gamma)^{n-1}u^{2k-1} \quad ,$$

i.e. provided

$$(p-\alpha-1)/(\beta-1) > k \text{ , and } p > k \text{ ,}$$

with

$$u^{p-\alpha-1-k(\beta-1)} \geq 2n\beta\epsilon^{\beta-1}(x-\gamma)^{n(\beta-1)-1}/(p-k) \quad ,$$

and

$$u^{p-k} \geq 4\epsilon nk(x-\gamma)^{n-1}/(p-k) \quad ,$$

which we call condition C' .

Condition D is satisfied provided

$$k < \alpha+k\beta \text{ , and } k < 2k-1 \text{ ,}$$

with

$$\tfrac{1}{2}n(n-1)\,u^{k}(x-\gamma)^{-2} \geq n\beta\epsilon^{\beta-1}(x-\gamma)^{n(\beta-1)-1}u^{\alpha+k\beta} \quad ,$$

and

$$\tfrac{1}{2}n(n-1)\,u^{k}(x-\gamma)^{-2} \geq 2\epsilon nk(x-\gamma)^{n-1}u^{2k-1} \quad ,$$

i.e. provided

$$\alpha+k(\beta-1) > 0 \text{ , and } k > 1 \text{ ,}$$

with

$$n(n-1)/\{2\beta\epsilon^{\beta-1}(x-\gamma)^{n(\beta-1)+1}\} \geq u^{\alpha+k(\beta-1)} \quad ,$$

and

$$(n-1)/\{4\epsilon k(x-\gamma)^{n+1}\} \geq u^{k-1} ,$$

which we call condition D' .

We conclude, therefore, that if

$$(p-\alpha-1)/(\beta-1) > k > 1, \quad p > k \text{ , and } \quad \alpha+k(\beta-1) > 0 , \tag{2.3.62}$$

then we may choose ϵ small enough to ensure that at least one of either C' or D' will always be true and inequality (2.3.61) is satisfied if $u_x^\beta = |u_x|^{\beta-1}u_x$, and $\beta > 1$.

Next, if $u_x^\beta = u_x$, i.e. if $\beta = 1$, then $S'/\epsilon d$ as defined by (2.3.60) becomes

$$\begin{aligned} S'/\epsilon d = -(p-k)u^{p+k-1} + n(x-\gamma)^{-1}u^{\alpha+k} - \epsilon\alpha(x-\gamma)^n u^{\alpha+k-1} \\ + 2\epsilon nk(x-\gamma)^{n-1}u^{2k-1} - n(n-1)(x-\gamma)^{-2}u^k, \end{aligned} \tag{2.3.63}$$

and will be less than or equal to zero if

$$(p-k)u^{p+k-1} + \frac{n(n-1)u^k}{(x-\gamma)^2} \geq \frac{nu^{\alpha+k}}{(x-\gamma)} + 2\epsilon nk(x-\gamma)^{n-1}u^{2k-1} . \tag{2.3.64}$$

It can be established, however, by arguments used in the proof of Theorem 2.3.4, that if

$$p > k > 1 ,$$

then ϵ may be chosen small enough to ensure that

$$\tfrac{1}{2}(p-k)u^{p+k-1} + \frac{\tfrac{1}{2}n(n-1)u^k}{(x-\gamma)^2} \geq 2\epsilon nk(x-\gamma)^{n-1}u^{2k-1} ,$$

for all possible u .

Hence, if ϵ is small and $p > k > 1$, inequality (2.3.64) will be satisfied if

$$\tfrac{1}{2}(p-k)\,u^{p+k-1} + \frac{\tfrac{1}{2}n(n-1)\,u^{k}}{(x-\gamma)^{2}} \geq \frac{nu^{\alpha+k}}{(x-\gamma)} \ . \tag{2.3.65}$$

This inequality, (2.3.65), will in turn be satisfied if either

$$\tfrac{1}{2}(p-k)\,u^{p+k-1} \geq \frac{nu^{\alpha+k}}{(x-\gamma)}$$

i.e. if

$$p > \alpha+1 \quad \text{and} \quad u^{p-(\alpha+1)} \geq \frac{2n}{(p-k)\,(x-\gamma)} \ , \tag{2.3.66}$$

or if

$$\frac{\tfrac{1}{2}n(n-1)\,u^{k}}{(x-\gamma)^{2}} \geq \frac{nu^{\alpha+k}}{(x-\gamma)} \ ,$$

i.e. if

$$u^{\alpha} \leq \frac{(n-1)}{2\,(x-\gamma)} \ . \tag{2.3.67}$$

Clearly, if $\alpha = 0$, then condition (2.3.67) is automatic for suitably large n .

Alternately, at least one of the conditions (2.3.66) or (2.3.67) will always be satisfied if

$$(n-1)^{\delta} > \frac{2^{1+\delta}n\,(x-\gamma)^{\delta-1}}{(p-k)} \tag{2.3.68}$$

where $\delta = \frac{p-\alpha-1}{\alpha} > 0$ by (2.3.66).

We assume further, however, that

$$\delta = \frac{p-\alpha-1}{\alpha} > 1 \ , \quad \text{i.e.} \quad p > 2\alpha+1 \tag{2.3.69}$$

in which case the requirement (2.3.68) may be satisfied for all $p > k$ and $x \in R_1$, provided n is chosen suitably large.

We conclude, therefore, that if $u_x^\beta = u_x$, then S' and hence S can be shown to be less than or equal to zero throughout R_1, provided

$$p > 2\alpha+1 \quad \text{and} \quad p > k > 1 \tag{2.3.70}$$

with n 'large' and ϵ 'small'.

Finally, if $u_x^\beta = |u_x|^\beta$, then (2.3.57) becomes

$$\begin{aligned} S' = -\epsilon d\{pu^{p-1}h - h'u^p\} + \epsilon^{\beta+1}d^{\beta+1}\alpha u^{\alpha-1}h^{\beta+1} \\ - \epsilon^\beta d^{\beta-1}d'\beta u^\alpha h^\beta + \epsilon^{\beta+1}d^{\beta+1}(\beta-1)u^\alpha h^\beta h' - \epsilon d''h + 2\epsilon^2 dd'hh'. \end{aligned} \tag{2.3.71}$$

If $h(u)$ and $d(x)$ are again defined by (2.3.59), then in this case S' will be less than or equal to zero if

$$\begin{aligned} (p-k)u^{p+k-1} + \frac{n(n-1)u^k}{(x-\gamma)^2} \geq (\alpha+k(\beta-1))\epsilon^\beta(x-\gamma)^{n\beta}u^{\alpha-1+k(\beta+1)} \\ + 2\epsilon nk(x-\gamma)^{n-1}u^{2k-1}. \end{aligned} \tag{2.3.72}$$

If we recall inequality (2.3.40) from the proof of Theorem 2.3.4, however, we see that it is exactly (2.3.72). The arguments used to establish (2.3.40) may therefore be applied directly here to show that (2.3.72) will be satisfied provided

$$(p-\alpha)/\beta > k > 1, \quad p > k \text{ , and } \quad \alpha + k\beta > 1 \text{ ,} \tag{2.3.73}$$

with n 'large' and ϵ 'small'. We conclude, therefore, that S' , and hence J , is less than or equal the zero as required, throughout R_1 , if in addition to condition (2.2.4) either

(a) $p > \alpha+\beta$, and $\beta \geq 1$, if $u_x^\beta = |u_x|^\beta$

(b) $p > \alpha+\beta$, and $\beta > 1$, if $u_x^\beta = |u_x|^{\beta-1}u_x$ or

(c) $p > 2\alpha+1$, if $u_x^\beta = u_x$.

To complete the proof of Lemma 2.3.5, as $J<0$ in R_1 , then

$$u_x(x,t) + \epsilon d(x) h(u(x,t)) < 0 \text{ , in } R_1 \text{ .} \tag{2.3.74}$$

If inequality (2.3.74) is integrated from γ to x_1 , for any $\gamma < x_1 < a$, as $u > 0$ in R_1 , and $h(u) = u^k$ with $k > 1$, then

$$\begin{aligned} u(x_1,t) &\leq \left\{(k-1)\int_\gamma^{x_1} \epsilon (x-\gamma)^n dx\right\}^{-1/(k-1)} \\ &= \{\epsilon(k-1)(x_1-\gamma)^{n+1}/(n+1)\}^{-1/(k-1)} \end{aligned}$$

for any $s_+ + ½\epsilon_0 = \gamma \leq x_1 \leq a$.

If γ_1 is now defined by $\gamma_1 = \gamma + ½\epsilon_0 = s_+ + \epsilon_0$, then

$$\begin{aligned} u(x,t) &\leq \{\epsilon(k-1)(\gamma_1-\gamma)^{n+1}/(n+1)\}^{-1/(k-1)} \\ &= C \end{aligned}$$

for any $\gamma_1 - s_+ + \epsilon_0 \leq x \leq a,\ T_0 \leq t \leq T$, and hence Lemma 2.3.5.

Remark 2.3.6

The proof of Lemma 2.3.5 has identified one result which does depend on the form chosen for the term u_x^β . From this proof we see that when $\beta - 1$, and u_x^β is chosen as $u_x^\beta - u_x$ then the assumption $p > 2\alpha+1$ is required in order to show that u is a bounded function to the right of $x - s(t)$, whereas the choice $u_x^\beta - |u_x|$ requires only the usual $p > \alpha+1$.

This can be compared with the results of Friedman & Lacey 1988, where this problem is considered in the case $\alpha - \beta - 1$ (so that $u^\alpha u_x^\beta - uu_x$) and the equivalent of Lemma 2.3.5 required the additional assumption $p > 3$ ($p > 2\alpha+1$ with $\alpha - 1$). We conclude, therefore, that in this case, the form chosen for u_x^β may play an important role in determining both the size and location of the blow-up sets for solutions to equations of this type.

We complete this subsection by showing that blow-up of the solution to (2.2.1)-(2.2.3) can occur at a single point.

We henceforth make the additional assumption on $\varphi(x)$, that

$$\varphi'' + \varphi^p - \varphi^\alpha(\varphi')^\beta \geq 0 \quad \text{for} \quad -a < x < a\ , \tag{2.3.75}$$

so that $u_t \geq 0$ at $t = 0$.

On differentiating (2.2.1) and setting $w(x,t) = u_t(x,t)$ it follows that

$$w_t = w_{xx} + pu^{p-1}w - \alpha u^{\alpha-1}|u_x|^{\beta-1}u_x w - \beta u^{\alpha}|u_x|^{\beta-1}w_x$$

for $-a < x < a,\ t > 0$

if $(u_x)^{\beta} = |u_x|^{\beta-1}u_x$,

and

$$w_t = w_{xx} + pu^{p-1}w - \alpha u^{\alpha-1}|u_x|^{\beta}w - \beta u^{\alpha}|u_x|^{\beta-1}sign(u_x)\,w_x$$

for $-a < x < a,\ t > 0$

if $(u_x)^{\beta} = |u_x|^{\beta}$.

Hence, as $u > 0$ throughout $(-a,a)$, and as $w = u_t > 0$ where $u_x = 0$ (because $u_x = 0$ within $(-a,a)$ at the unique local maximum $x = s(t)$), the maximum principle applied to u_t establishes that

$$u_t > 0 \quad \text{if} \quad -a < x < a,\ 0 < t < T . \qquad (2.3.76)$$

Theorem 2.3.7

If u is the solution to (2.2.1)-(2.2.3), with φ satisfying (2.2.5), (2.3.1)-(2.3.3), and (2.3.75) and if the conditions of Theorem 2.3.4 and Lemma 2.3.5 are satisfied, then u blows up at a single point.

Proof

The conditions of Theorem 2.3.4 and Lemma 2.3.5 can be summarised by requiring, in addition to (2.2.4), that:-

(i) $p > \alpha+\beta$ and $\beta \geq 1$, if $u_x^\beta = |u_x|^\beta$,

(ii) $p > \alpha+\beta$ and $\beta > 1$, if $u_x^\beta = |u_x|^{\beta-1}u_x$,

(iii) $p > 2\alpha+1$ if $u_x^\beta = u_x$,

and in light of these results, it suffices to show that $s_+ = s_-$.

By definition, $s_+ \geq s_-$.

We assume that

$$s_+ > s_- , \tag{2.3.77}$$

and let

$$s_1 = s_- - \epsilon,\ s_2 = s_+ - \epsilon \text{ , and } \delta = (s_1+s_2)/2 \tag{2.3.78}$$

where $3\epsilon < s_+ - s_-$.

We consider a region

$$R = \{ s_1 < x < s_2,\ T_0 < t < T \} , \tag{2.3.79}$$

where T_0 is chosen close enough to T so that

$$s(T_0) > s_2 , \tag{2.3.80}$$

and

$$u(s_1, t) < u(s_2, t) \quad \text{if} \quad T_0 < t < T \ . \tag{2.3.81}$$

Here we are the fact that u is increasing in t (from (2.3.75)), and hence that

$$u(s_2, t) \to +\infty \quad \text{as} \quad t \to T \tag{2.3.82}$$

(from (2.3.78)), and

$$u(s_1, t) \le C < \infty \quad \text{if} \quad 0 < t < T \tag{2.3.83}$$

(from (2.3.78) and Theorem 2.3.4).

We next introduce the function

$$w(x, t) = u(x, t) - v(x, t) \tag{2.3.84}$$

where

$$v(x, t) = u(2\delta - x, t)$$

and we try to show that $w > 0$, in $R \cap \{ x > \delta \}$. From (2.3.84), (2.3.80) and (2.3.81), we see that

$$w(\delta, t) = u(\delta, t) - u(\delta, t) = 0 \ ,$$

$$w(s_2, t) = u(s_2, t) - u(s_1, t) > 0 \ ,$$

and $$w(x, T_0) = u(x, T_0) - v(x, T_0) > 0 \ ,$$

if T_0 is close enough to T .

Hence, $w > 0$ on the parabolic boundary of $R \cap \{ x > \delta \}$. We now proceed, as in the proof of Lemma 2.3.2, to show that $w > 0$ in $R \cap \{ x > \delta \}$. On differentiating with respect to t we find that w satisfies

$$w_t = w_{xx} + u^p - v^p - u^\alpha u_x^\beta + v^\alpha (-v_x)^\beta \ ,$$

i.e. the same equation as in that proof. Hence, on following the same argument as used there, we are able to show that

$$w > 0 \ , \text{in} \ R \cap \{ x > \delta \} \ .$$

It follows, consequently, that

$$s(t) > \delta \ , \text{for} \ T_0 < t < T \ ,$$

and hence

$$s_- = \liminf_{t \to T} s(t) \geq \delta = (s_1 + s_2)/2 = (s_+ + s_-)/2 - \epsilon \ ,$$

so that

$$s_- \geq s_+ - 2\epsilon \ ,$$

which, if ϵ is sufficiently small, is a contradiction to (2.3.77).

We conclude that $s_+ = s_-$ and blow up occurs at the single point

$$s_+ = s_- = \lim_{t \to T} s(t) \ .$$

Section 2.4 Estimate of blow-up rate for the negative gradient case

The preceding subsections have considered the question of when the solution to a problem such as (2.2.1)-(2.2.3) will exhibit finite-time blow-up, and when this blow-up will occur at a single point. In this section we take this analysis one stage further by establishing some estimates on the rate at which single-point blow-up takes place.

We consider a solution to problem (2.2.1)-(2.2.3) which satisfies the requirements of Theorem 2.3.7, and which does, therefore, blow up at a single point, at the finite time T, say.

To summarise the conditions required by Theorem 2.3.7, $u(x,t)$ is the solution to

$$u_t = u_{xx} + u^p - u^\alpha u_x^\beta \quad \text{in} \quad (-a,a),\ t>0 \tag{2.2.1}$$

$$u(x,0) = \varphi(x) \quad \text{in} \quad (-a,a) \tag{2.2.2}$$

$$u(\pm a,t) = 0 \quad \text{for} \quad t>0\ , \tag{2.2.3}$$

with (minimally)

$$p > 1,\ \beta \geq 1 \quad \text{and} \quad \alpha \geq 0\ . \tag{2.2.4}$$

Further, $\varphi(x)$ must be assumed 'large enough' (from Theorem 2.2..3) with

$$\varphi \in C^1[-a,a],\quad \varphi \geq 0,\quad \varphi(\pm a) = 0\ , \qquad \text{(2.2.5) and (2.3.1)}$$

and there is assumed to exist some $x_0 \in (-a,a)$ such that

$$\varphi'(x) \geq 0 \quad \text{if} \quad -a \leq x \leq x_0, \; \varphi'(x) \leq 0 \quad \text{if} \quad x_0 \leq x \leq a \ , \tag{2.3.2}$$

and where

$$\varphi(x) \geq \varphi(-x) \quad \text{for} \quad x \geq 0 \ . \tag{2.3.3}$$

Finally, Theorem 2.3.7 requires, in addition to (2.2.4), that the parameters α, β and p satisfy the following relationships for the stated form of the function u_x^β ,

(i) if $u_x^\beta = |u_x|^\beta$ then $p > \alpha+\beta$,

(ii) if $u_x^\beta = |u_x|^{\beta-1}u_x$ then $\beta > 1$ requires $p > \alpha+\beta$,

and (iii) if $u_x^\beta = u_x$ then $p > 2\alpha+1$.

We proceed by considering the function $m(t)$, as defined in Section 2.2.1, where

$$m(t) = \max_{x \in (-a,a)} u(x,t) \ , \tag{2.4.1}$$

and establish the following Theorem.

Theorem 2.4.1

The function $m(t)$ is Lipschitz continuous, and

$$m'(t) \leq m^p(t) \tag{2.4.2}$$

at any point at which m is differentiable.

Proof

The proof of this result is identical to that of Theorem 4.5 from Friedman & MacLeod 1985 and is established as follows:-

Let $m(t_i) = u(x_i, t_i)$, $i = 1,2$ and set $h = t_2 - t_1$.

Then

$$\begin{aligned} m(t_2) - m(t_1) &= u(x_2, t_2) - u(x_1, t_1) \\ &\geq u(x_1, t_2) - u(x_1, t_1) \\ &= hu_t(x_1, t_1) + o(h) . \end{aligned}$$

Similarly,

$$\begin{aligned} m(t_2) - m(t_1) &= u(x_2, t_2) - u(x_1, t_1) \\ &\leq u(x_2, t_2) - u(x_2, t_1) \\ &= hu_t(x_2, t_2) + o(h) . \end{aligned}$$

It follows, therefore, that $m(t)$ is Lipschitz continuous.

In addition, if $t_2 > t_1$, then

$$m(t_2) - m(t_1) \leq (t_2 - t_1)\, u_t(x_2, t_2) + o(h)$$

so that $$\frac{m(t_2) - m(t_1)}{t_2 - t_1} \leq u_t(x_2, t_2) + o(1) .$$

Hence, as $u_{xx}(x_2, t_2) \leq 0$, and $u_x(x_2, t_2) = 0$, it follows that

$$\begin{aligned} u_t(x_2, t_2) &= \left\{ u_{xx} + u^p - u^\alpha u_x^\beta \right\} (x_2, t_2) \\ &\leq u^p(x_2, t_2) \\ &= m^p(t_2), \end{aligned}$$

and therefore that

$$m' \leq m^p , \tag{2.4.3}$$

at any point of differentiability of the function $m(t)$. Integrating (2.4.3) from t to T , and using the fact that $m(T) = +\infty$, then yields the following Corollary.

Corollary 2.4.2

If $u(x,t)$ is a solution to problem (2.2.1)-(2.2.3) which satisfies the requirements of Theorem 2.3.7 (summarised in the introduction to this section), then there exists a constant $c > 0$ such that

$$m(t) \geq \frac{C}{(T-t)^{1/p-1}} \tag{2.4.4}$$

for any $0 < t < T$.

We next assume, in addition to (2.2.4) and (2.3.1)-(2.3.3), that $\varphi(x)$ may be chosen such that

$$\varphi_{xx} + \varphi^{p} - \varphi^{\alpha}\varphi_x^{\beta} \geq 0 \quad \text{in} \quad (-a,a) \tag{2.4.5}$$

for either choice of the term φ_x^{β} (i.e. $\varphi_x^{\beta} = |\varphi_x|^{\beta}$ or $\varphi_x^{\beta} = |\varphi_x|^{\beta-1}\varphi_x$).

If so, then $u_t(x,0) \geq 0$ in $(-a,a)$ by (2.4.5) and, as $u_t = 0$ for $x = \pm a,\ t > 0$, it follows that $u_t \geq 0$ on the parabolic boundary of $(-a,a)x(0,T)$.

Next, on differentiating (2.2.1) with respect to t, and setting

$w(x,t) = u_t(x,t)$, we see that

$$w_t = w_{xx} + pu^{p-1}w - \alpha u^{\alpha-1}u_x^{\beta}w - \beta u^{\alpha}u_x^{\beta-1}w_x \quad \text{in} \quad (-a,a),\ t>0\ . \tag{2.4.6}$$

Further, as $u = 0$ only for $x = \pm a$, and as $\beta \geq 1$ by (2.2.4), then there exist constants c and d, bounded throughout the interior of $(-a,a)x(0,T)$, for which

$$w_t = w_{xx} + cw + dw_x \quad \text{in} \quad (-a,a),\ t>0\ . \tag{2.4.7}$$

The maximum principle applied to (2.4.7) consequently yields the conclusion that

$$w = u_t > 0 \quad \text{within} \quad (-a,a)x(0,T)\ . \tag{2.4.8}$$

We now derive a 'better' lower bound for u_t away from the parabolic boundary of this set.

Taking Ω_T to be $(-a,a)x(0,T)$, we define the set Ω^{η}, for any $\eta > 0$, as

$$\Omega^{\eta} = \{-a+\eta < x < a-\eta\}\ , \tag{2.4.9}$$

and the function $J(x,t)$ as

$$J(x,t) = u_t(x,t) - c(t)g(u(x,t))\ . \tag{2.4.10}$$

We choose

$$c(t) = \epsilon e^{-Mt} \tag{2.4.11}$$

for some small $\epsilon > 0$, and positive constant M and

$$g(u) = u^{s} \tag{2.4.12}$$

for some $s > 0$.

In light of Theorem 2.3.7, u blows up at a single interior point of Ω_T ,

whereby, if η is small enough, it follows that

$$g(u) = u^{s} \leq c_0 < \infty \quad \text{for} \quad x = \pm(a-\eta)\,,\ 0 < t < T\ .$$

Further, we see from (2.4.8) that

$$u_t \geq c > 0$$

on the parabolic boundary of the set $\Omega^{\eta}x(\eta, T)$ and hence, that if ϵ is chosen small enough then

$$J(x,t) = u_t(x,t) - \epsilon e^{-Mt}u^{s}(x,t) \tag{2.4.13}$$

will be strictly greater than zero as on the parabolic boundary of $\Omega^{\eta}x(\eta, T)$.

Differentiating (2.4.10) we see that

$$J_t = u_{tt} - c'g - cg'u_t \tag{2.4.14}$$

$$J_x = u_{tx} - cg'u_x \tag{2.4.15}$$

and $$J_{xx} = u_{txx} - cg''u_x^2 - cg'u_{xx} \tag{2.4.16}$$

so that

$$J_t - J_{xx} = pu^{p-1}u_t - \alpha u^{\alpha-1}u_x^{\beta}u_t - \beta u^{\alpha}u_x^{\beta-1}u_{xt}$$
$$- c'g + cg''u_x^2 - cg'u^p + cg'u^{\alpha}u_x^{\beta}.$$
(2.4.17)

Using equations (2.4.10) and (2.4.15) to substitute for u_t and u_{xt} in the right hand side of (2.4.17) yields

$$J_t - J_{xx} + \beta u^{\alpha}u_x^{\beta-1}J_x - \{pu^{p-1} - \alpha u^{\alpha-1}u_x^{\beta}\}J = S \tag{2.4.18}$$

where

$$S = c\,\{pu^{p-1}g - g'u^p\} - \alpha u^{\alpha-1}u_x^{\beta}cg$$
$$- (\beta-1)u^{\alpha}u_x^{\beta}cg' - c'g + cg''u_x^2 \ . \tag{2.4.19}$$

Further, as $\beta \geq 1$ by (2.2.4), and as $u > 0$ throughout $\Omega \cap X(\eta, T)$, each of the coefficients on the left hand side of (2.4.18) must necessarily remain bounded within this set.

On substituting for $c(t)$ and $g(u)$, (2.4.19) becomes

$$S = \epsilon e^{-Mt}\Big\{(p-s)\,u^{p+s-1} - (\alpha+s(\beta-1))\,u^{\alpha+s-1}u_x^{\beta}$$
$$+ Mu^{s} + s(s-1)\,u^{s-2}u_x^2\Big\}$$

whereby

$$S \geq \epsilon e^{-Mt}\Big\{(p-s)\,u^{p+s-1} - (\alpha+s(\beta-1))\,u^{\alpha+s-1}\,|\,u_x\,|^{\beta}$$
$$+ Mu^{s} + s(s-1)\,u^{s-2}u_x^2\Big\}.$$
(2.4.20)

If it can be established that S as defined by (2.4.20) is no less than zero throughout $\Omega \cap X(\eta, T)$, then by applying the maximum principle to (2.4.18), and as we have already demonstrated that $J(x, t)$ is strictly greater than zero on the parabolic boundary of this set, we may conclude that J is strictly greater than zero throughout $\Omega^\eta x(\eta, T)$.

It is necessary, therefore, to investigate the sign of the expression S . We shall consider three cases, in each of which we seek to demonstrate that S , as defined by (2.4.20), is no less than zero throughout $\Omega^\eta x(\eta, T)$.

Case 1 $\quad \beta < 2(p-\alpha)/(p+1)$.

It is immediate from (2.4.20) that S will be greater than or equal to zero if

$$s(s-1)u^{s-2}u_x^2 \geq (\alpha+s(\beta-1))u^{\alpha+s-1}|u_x|^\beta , \tag{2.4.21}$$

and if

$$p \geq s > 1 .$$

If inequality (2.4.21) fails, however, i.e. if

$$|u_x|^{2-\beta} \leq \left\{\frac{\alpha+s(\beta-1)}{s(s-1)}\right\} u^{\alpha+1} \tag{2.4.22}$$

then provided $\beta < 2$, (2.4.22) may be used to estimate $|u_x|^\beta$ in (2.4.20), upon which we see that S will be of the desired sign if

$$(p-s)u^{p+s-1} + Mu^s \geq Du^{\alpha+s-1+\beta(\alpha+1)/(2-\beta)} \tag{2.4.23}$$

where

$$D = \frac{(\alpha+s(\beta-1))^{1+\frac{\beta}{2-\beta}}}{(s(s-1))^{\frac{\beta}{2-\beta}}} \ . \tag{2.4.24}$$

Inequality (2.4.23) will clearly be satisfied if

$$(p-s)\,u^{p+s-1} \geq D u^{\alpha+s-1+\beta(\alpha+1)/(2-\beta)}$$

which we call condition E, or if

$$Mu^{s} \geq D u^{\alpha+s-1+\beta(\alpha+1)/(2-\beta)}$$

which we call condition F.

Condition E will be satisfied for $p > s$ if

$$p+s-1 > \alpha+s-1+\beta(\alpha+1)/(2-\beta)$$

which is automatic for $\beta < 2(p-\alpha)/(p+1)$, and if u is 'large enough' compared to D.

Alternately, condition F will be satisfied if

$$s \leq \alpha+s-1+\beta(\alpha+1)/(2-\beta) \ , \tag{2.4.25}$$

i.e. if $\alpha+\beta \geq 1$, which is also automatic in light of (2.2.4), and if either u is small enough compared to M, or simply if M is large enough compared to D (if equality in (2.4.25)).

We conclude that at least one of conditions E or F will be true if M is chosen sufficiently large, and hence that s, as defined by (2.4.19) will be greater than or equal to zero as required for any $p > s > 1$ in the case

$$\beta < 2(p-\alpha)/(p+1) \ .$$

Case 2 $\qquad \beta = 2(p-\alpha)/(p+1)$.

If $\beta = 2(p-\alpha)/(p+1)$, then the arguments of Case 1 apply directly to the point where we observe that s will be of the desired sign if either condition E , that

$$(p-s)\,u^{p+s-1} \geq Du^{\alpha+s-1+\beta(\alpha+1)/(2-\beta)} \ ,$$

or condition F , that

$$Mu^{s} \geq Du^{\alpha+s-1+\beta(\alpha+1)/(2-\beta)} \ ,$$

hold, where

$$D = \frac{(\alpha+s(\beta-1))^{1+\beta/(2-\beta)}}{(s(s-1))^{\beta/(2-\beta)}} \ .$$

($\beta < 2$ is automatic for $\beta = 2(p-\alpha)/(p+1)$ in light of (2.2.4))

In this case, however, as $\beta = 2(p-\alpha)/(p+1)$, then

$$p+s-1 = \alpha+s-1+\beta(\alpha+1)/(2-\beta)$$

so that the powers of u on both sides of the inequality required by condition E are identical. Condition E will only be satisfied, therefore, if

$$(p-s) \geq D \ ,$$

i.e. if

$$p-s \geq \frac{(\alpha+s(\beta-1))^{1+\beta/(2-\beta)}}{(s(s-1))^{\beta/(2-\beta)}} . \tag{2.4.26}$$

For $\beta = 2(p-\alpha)/(p+1)$, it follows that

$$\beta/(2-\beta) = (p-\alpha)/(\alpha+1) ,$$

$$1+\beta/(2-\beta) = (p+1)/(\alpha+1) ,$$

and $\beta-1 = \dfrac{p-(2\alpha+1)}{p+1}$,

so that (2.4.26) is equivalent to

$$(p-s)\{s(s-1)\}^{\frac{p-\alpha}{\alpha+1}} \geq \left\{\alpha + \frac{s(p-(2\alpha+1))}{p+1}\right\}^{\frac{p+1}{\alpha+1}} . \tag{2.4.27}$$

We therefore choose

$$s = (p-\alpha)/\beta = \tfrac{1}{2}(p+1)$$

in which case $p > s > 1$ is automatic for $p > 1$, and (2.4.27) becomes

$$\tfrac{1}{2}(p-1)\{\tfrac{1}{2}(p+1)\tfrac{1}{2}(p-1)\}^{\frac{p-\alpha}{\alpha+1}} \geq (\tfrac{1}{2}(p-1))^{\frac{p+1}{\alpha+1}}$$

which reduces to

$$\tfrac{1}{2}(p+1) \geq 1 \quad \text{for} \quad p > 1 ,$$

and is automatic in light of (2.2.4).

We conclude, therefore, that s , as defined by (2.4.19) will be greater than or equal to zero as required if $s = (p-\alpha)/\beta = \tfrac{1}{2}(p+1)$, for $\beta = 2(p-\alpha)/(p+1)$.

Case 3

It remains, finally, to examine the sign of the right hand side of inequality (2.4.20) in the case that

$$\beta > 2(p-\alpha)/(p+1) \quad . \tag{2.4.28}$$

The right hand side of (2.4.20) will be no less than zero if

$$(p-s)u^{p+s-1} + Mu^{s} + s(s-1)u^{s-2}u_x^2 \geq (\alpha+s(\beta-1))u^{\alpha+s-1}|u_x|^{\beta} \quad . \tag{2.4.29}$$

As inequality (2.4.28) holds, it is possible in this case to make use of the upper estimate for $|\nabla u|$ developed in Appendix A. Appendix A shows that

$$|\nabla u|^2 - |u_x|^2 \leq \{Cu^{m} + u^{k} - Au^{l} + B\}^2 \tag{2.4.30}$$

for appropriate constants A, B and C, provided

(i) $\beta > 2(p-\alpha)/(p+1)$, i.e. (2.4.28)

(ii) $C^{\beta}\{\alpha+n(\alpha+m(\beta-1))\} > p$

(iii) $C \geq 1$ and $m - (p-\alpha)/\beta > k > l > 1$

where k is 'close enough' to m and n is the largest integer such that $n \leq \beta$.

If (2.4.30) is used to estimate $|u_x|^{\beta}$ in the right hand side of (2.4.29) then this inequality will be satisfied if

$$\begin{aligned}(p-s)u^{p+s-1} &+ Mu^{s} + s(s-1)u^{s-2}u_x^2 \\ &\geq (\alpha+s(\beta-1))u^{\alpha+s-1}|Cu^{m}+u^{k}-Au^{l}+B|^{\beta}\end{aligned}$$

which is clearly automatic if

$$(p-s)u^{p+s-1} \geq (\alpha+s(\beta-1))u^{\alpha+s-1}|Cu^{m}+u^{k}+B|^{\beta} \tag{2.4.31}$$

which we call condition E' .

As $m = (p-\alpha)/\beta$, however, and we assume $p>s$, then the highest power of u in the right hand side of (2.4.29) is also $p+s-1$. Condition E' will only be satisfied for 'large' u therefore, if

$$(p-s) \geq C^{\beta}(\alpha+s(\beta-1)) \tag{2.4.32}$$

and if u is 'large enough' compared to C, A and B .

We assume for the moment therefore that condition (2.4.32) is not incompatible with the requirements (labelled (i)-(iii)) of the estimate (2.4.30).
Alternately, inequality (2.4.29) will also be satisfied if

$$Mu^{s} + s(s-1)u^{s-2}u_x^2 \geq (\alpha+s(\beta-1))u^{\alpha+s-1}|u_x|^{\beta}$$

which may be expressed as

$$\begin{aligned}\{ (s(s-1)^{1/2}u^{1/2(s-2)}|u_x| - \tfrac{1}{2}(s(s-1))^{-1/2}(\alpha+s(\beta-1))u^{\alpha+1/2 s}|u_x|^{\beta-1} \}^2 \\ + Mu^{s} \geq \tfrac{1}{4}(s(s-1))^{-1}(\alpha+s(\beta-1))^2u^{2\alpha+s}|u_x|^{2(\beta-1)}\end{aligned} \tag{2.4.33}$$

and will be satisfied if

$$Mu^{s} \geq \tfrac{1}{4}(s(s-1))^{-1}(\alpha+s(\beta-1))^2u^{2\alpha+s}|u_x|^{2(\beta-1)} \ . \tag{2.4.34}$$

As (2.2.4) ensures that $\beta \geq 1$, we may again use (2.4.30) to estimate $|u_x|^{2(\beta-1)}$ in the right hand side of (2.4.34), upon which we see that this inequality will be satisfied if

$$Mu^{s} \geq \tfrac{1}{4}(s(s-1))^{-1}(\alpha+s(\beta-1))^{2}u^{2\alpha+s}\,|\,Cu^{m}+u^{k}+B\,|^{2(\beta-1)}$$

which we call condition F'.

Condition F' is consequently dependent on the requirements of the estimate (2.4.30).

Assuming these requirements hold, condition F' is satisfied provided $\alpha \geq 0$, and u is 'small enough' compared to M.

It follows, that if the condition (2.4.32) can be satisfied in addition to the requirements (i)-(iii) of the estimate (2.4.30), then by choosing M sufficiently large we may ensure that at least one of conditions E' or F' will always be satisfied, and hence that the right hand side of inequality (2.4.20) is no less than zero as required.

It remains to verify that the conditions (i)-(iii) of estimate (2.4.30), i.e.

(i) $\beta > 2(p-\alpha)/(p+1)$

(ii) $C^{\beta}\{\alpha+n(\alpha+m(\beta-1))\} > p$

(iii) $C \geq 1$ and $m - (p-\alpha)/\beta > k > l > 1$

where k is 'close enough' to m and n is the largest integer such that

$n \leq \beta$,

and condition (2.4.32),

$$(p-s) \geq C^{\beta}(\alpha+s(\beta-1)) \quad ,$$

are compatible, for some $p > s > 1$.

The requirements (ii) and (iii) above will always be satisfied if

$$C^{\beta} \geq D - \max\left[\, 1,\ p(\alpha+n(\alpha+m(\beta-1))^{-1}\, \right] \, . \tag{2.4.35}$$

We first assume that $D - 1$, i.e. that

$$p \leq \alpha + n(\alpha+m(\beta-1)) \quad , \tag{2.4.36}$$

in which case there exists $p > s > 1$ to satisfy (2.4.32) provided

$$p-s > (\alpha+s(\beta-1))C^{\beta} - \alpha+s(\beta-1)$$

i.e. provide

$$(p-\alpha)/\beta > s > 1$$

which is automatic for $p > \alpha+\beta$.

It follows therefore that these conditions are compatible if $D - 1$.

Alternatively, if $D > 1$ i.e. if

$$p > \alpha + n(\alpha+m(\beta-1)) \quad , \tag{2.4.37}$$

then (2.4.32) requires that there exits $p > s > 1$ such that

$$\frac{(p-s)}{\alpha+s(\beta-1)} > C^{\beta} \geq D - \frac{p}{\alpha + n(\alpha+m(\beta-1))}$$

where $m = (p-\alpha)/\beta$ and n is the largest integer satisfying $n \le \beta$.

A suitable s will exist, therefore, provided

$$pn(\alpha+m(\beta-1)) > \alpha(n+1) + (\beta-1)(nm+p)$$

i.e. if

$$p > 1 + \frac{\{\alpha+p(\beta-1)\}}{n(\alpha+m(\beta-1))} \tag{2.4.38}$$

and will satisfy the inequality

$$\frac{pn(\alpha+m(\beta-1))}{\alpha(n+1) + (\beta-1)(nm+p)} > s \ . \tag{2.4.39}$$

Hence, as $m = (p-\alpha)/\beta$, it follows that $\alpha+m(\beta-1) = p-m$, whereby, on adding and subtracting $m(\beta-1)$ to the top line of the fractional term in (2.4.38), this inequality becomes

$$p > 1 + \frac{1}{n} + \frac{(\beta-1)}{n} = 1 + \beta/n \ . \tag{2.4.40}$$

We conclude, therefore, that these necessary requirements are also compatible in the case of (2.4.37) provided condition (2.4.40) is satisfied. In this case, s will be bounded from above as described in (2.4.39).

In summary, we see that satisfaction of the inequality (2.4.29) in the case $\beta > 2(p-\alpha)/(p+1)$ may be considered in two circumstances.

Case A

If D (as defined by (2.4.35)) takes the value 1 , i.e. if

$$p \le \alpha + n(\alpha+m(\beta-1))$$

where $m - (p-\alpha)/\beta$ and n is the largest integer such that $n \le \beta$, then a suitable value of s exists to satisfy inequality (2.4.29), and

$$(p-\alpha)/\beta > s > 1 .$$

Case B

Alternatively, if $D > 1$, i.e. if

$$p > \alpha + n(\alpha+m(\beta-1))$$

then a suitable value of s exists to satisfy (2.4.29) only if

$$p > 1+\beta/n ,$$

where again $m - (p-\alpha)/\beta$ and n is the largest integer such that $\beta \ge n$, whereby

$$\frac{pn(\alpha+m(\beta-1))}{\alpha(n+1) + (\beta-1)(nm+p)} > s > 1 .$$

Remark

The inequality (2.4.40) is automatic in light of (2.2.4) and the conditions of Theorem 2.3.7 (the satisfaction of which is assumed throughout this section), if $\alpha \ge 1$, $\beta \ge 2$, or $p \ge 3$.

Proof

Theorem 2.3.7 requires that $p > \alpha+\beta$. Hence, if $\alpha \ge 1$, then

$$p > 1+\beta .$$

From (2.2.4), $\beta \geq 1$ which means that n, the largest integer satisfying $n \leq \beta$, also satisfies $n \geq 1$. It follows that

$$p > \alpha+\beta \geq 1+\beta \geq 1+\beta/n \ .$$

If $\beta \geq 2$, then again we have that $n \geq 2$, and

$$p - (1+\beta/n) \geq p - (1+\beta/2) > \beta/2 - 1 \geq 0$$

as $p > \alpha + \beta \geq \beta \geq 2$.

Finally, as $\beta = n + \delta$ for some $0 \leq \delta < 1$, we see that

$$1 + \beta/n = 1 + \frac{(n+\delta)}{n} = 2 + \delta/n = 2 + \delta < 3$$

and $p \geq 3$ will ensure that $p > 1 + \beta/n$ as required.

We conclude, therefore, that the right hand side of (2.4.20) will be no less than zero in each of the cases 1, 2 or 3. In each of these cases, it follows, as proposed, from the maximum principle applied to (2.4.18), that $J(x,t)$, as defined by (2.4.13), is strictly greater than zero throughout the set $\Omega^{\eta} \times (\eta, T)$, i.e. that

$$J(x,t) = u_t(x,t) - \epsilon e^{-Mt} u^s(x,t) > 0 \quad \text{in} \quad \Omega^{\eta} \times (\eta, T) \tag{2.4.41}$$

for suitable values of $s > 1$, and provided ϵ and $1/M$ are sufficiently small.

Before proceeding to describe the consequences of inequality (2.4.41), we first re-state, in detail, the values of the parameters α, β, p and s, for which this inequality has been verified.

The function $J(x,t)$ defined by (2.4.13) is considered for all α, β and p for which Theorem 2.3.7 applies, and these requirements are summarised at the beginning of this section. Inequality (2.4.41) has been verified in each of the cases 1, 2 and 3 described, and the requirements of these conditions are summarised as follows:-

Case 1

If $\beta < 2(p-\alpha)/(p+1)$, then inequality (2.4.41) is satisfied for any s, such that

$$p \geq s > 1 .$$

Case 2

If $\beta = 2(p-\alpha)/(p+1)$, then inequality (2.4.41) is satisfied for

$$s = (p-\alpha)/\beta = \tfrac{1}{2}(p+1) > 1 .$$

Case 3

If $\beta > 2(p-\alpha)/(p+1)$ then inequality (2.4.41) does not apply for all considered α, β and p. It has however been verified in the following cases.

(A) If, in addition to $\beta > 2(p-\alpha)/(p+1)$, we have that

$$p \leq \alpha+n(\alpha+m(\beta-1))$$

for $m = (p-\alpha)/\beta$ and n the largest integer such that $n \le \beta$, then inequality (2.4.41) is verified for

$$(p-\alpha)/\beta > s > 1 .$$

(B) Alternatively, if

$$p > \alpha+n(\alpha+m(\beta-1)) ,$$

then inequality (2.4.41) requires in addition that

$$p > 1+\beta/n ,$$

and is verified for s satisfying the estimate (2.4.39). After suitable manipulation, however, this inequality reduces to

$$\frac{pn}{n+\beta} > s > 1$$

for n the largest integer such that $\beta \ge n$.

In each of the cases considered, inequality (2.4.41) may be integrated from t to T , whereby, as $u > 0$ in $\Omega^{\eta}x(\eta,T)$, and as $s > 1$ in each of these cases, it follows that there exists a constant $C > 0$ for which

$$u(x,t) \le \frac{C}{(T-t)^{1/(s-1)}} \quad \text{for} \quad (x,t) \quad \text{in} \quad \Omega^{\eta}x(\eta,T) .$$

It is also true, however, in light of Theorem 2.3.7, that if η is sufficiently small, then $u(x,t)$ is also bounded at any other points in Ω_T, and we have therefore established the following Theorem.

Theorem 2.4.3

If $u(x,t)$ is a solution to problem (2.2.1) - (2.2.3) which satisfies the requirements of Theorem 2.3.7 (summarised at the beginning of this section) then u blows up at a finite time, say T. If, in addition, the initial value $\varphi(x)$ satisfies condition (2.4.5) and if the parameters α, β and p are suitably related, then there exist constants $0 < C_1 < \infty$, and $s > 1$ such that

$$u(x,t) \le \frac{C_1}{(T-t)^{1/(s-1)}} \quad \text{for} \quad (x,t) \in \Omega_T. \tag{2.4.42}$$

Constants C_1 and $s > 1$ can be found to satisfy inequality (2.4.42) if, in addition to the requirements of Theorem 2.3.7, α, β and p satisfy at least one of the following three conditions,

(i) $\beta \le 2(p-\alpha)/(p+1)$,

(ii) $\beta > 2(p-\alpha)/(p+1)$ and $p \le \alpha + n(\alpha+m(\beta-1))$

or (iii) $\beta > 2(p-\alpha)/(p+1), \beta > \alpha + n(\alpha+m(\beta-1))$ and $p > 1 + \beta/n$

where $m = (p-\alpha)/\beta$ and n is the largest integer satisfying $n \le \beta$.

A range of values of s for which inequality (2.4.42) holds in each of the cases (i)-(iii) is described by Cases 1-3 of page 214 which are recalled below.

Case 1.

If $\beta < 2(p-\alpha)/(p+1)$, then inequality (2.4.42) holds for any s such that

$$p \geq s > 1.$$

Case 2.

If $\beta - 2(p-\alpha)/(p+1)$, then inequality (2.4.42) holds for

$$s - (p-\alpha)/\beta - \tfrac{1}{2}(p+1) > 1.$$

Case 3.

If $\beta > 2(p-\alpha)/(p+1)$, then the inequality (2.4.42) does not hold for all α, β and p considered. It has however been verified in the following circumstances.

(A) If, in addition to $\beta > 2(p-\alpha)/(p+1)$, we have that

$$p \leq \alpha + n(\alpha+m(\beta-1))$$

for $m - (p-\alpha)/\beta$ and n the largest integer such that $n \leq \beta$, then inequality (2.4.42) holds for

$$(p-\alpha)/\beta > s > 1.$$

(B) Alternately, if

$$p > \alpha + n(\alpha+m(\beta-1))$$

then inequality (2.4.42) requires in addition that

$$p > 1 + \beta/n,$$

and will hold for values of s satisfying inequality (2.4.39). After suitable manipulation, inequality (2.4.39) reduces to,

$$\frac{pn}{n+\beta} > s > 1$$

for n the largest integer such that $n \le \beta$.

Remark

If either $\alpha \ge 1$, $\beta \ge 2$ or $p \ge 3$ then the requirement (in Case 3 (B)) that $p > 1 + \beta/n$ can be shown to be automatic. Hence, in either of these cases, no additional requirements need be made of the relative sizes of α, β and p other than those already necessary to apply Theorem 2.3.7.

Proof

(a) If $\alpha \ge 1$, then $\beta \ge 1$ ensures that $n \ge 1$, and as $p > \alpha + \beta$ is a requirement of Theorem 2.3.7,

$$p > \alpha+\beta \ge 1+\beta \ge 1+\beta/n.$$

(b) If $\beta \ge 2$ then $n \ge 2$ so that

$$p-1-\beta/n \ge p-1-\beta/2 > p/2-1 > 0$$

and $p > 1+\beta/n$

as $p > \alpha+\beta$ (from Theorem 2.3.7), so that $p > \alpha+\beta \geq \beta \geq 2$.

(c) Finally, $\beta = n+\delta$ for some $0 \leq \delta < 1$.

Hence,

$$1+\beta/n = 1+\frac{(n+\delta)}{n} = 2+\delta/n < 3,$$

and $p \geq 3$ ensures $p > 1+\beta/n$ as proposed.

Section 2.5 Existence of blow-up for a positive gradient term

Sections 2.2-2.4 have considered the problem (2.2.1)-(2.2.3) in which the gradient term is always of negative sign. Initially, an equation in which the gradient term is wholly positive would appear less interesting, as a number of results on finite time blow-up of the solution are immediately available (e.g. Theorem 2.5.1 below).

In Section 2.3, however, where the problem of identifying the size and location of the blow-up sets is addressed, we have found cases where it is the positive nature of the gradient term which can determine the amount of information we are able to obtain about these regions.

This section consequently seeks to extend the methods of Section 2.3, for determining when single-point blow-up can take place, to equations of this type.

Before presenting this analysis, we first note that if the gradient term has positive sign, but is an odd function of the gradient, i.e. if $u(x,t)$ satisfies

$$u_t = u_{xx} + u^p + u^\alpha |u_x|^{\beta-1} u_x$$

then this solution will, for appropriately chosen initial condition $\varphi(x)$, satisfy all the results of Sections 2.2-2.4.

To see this we define the function $v(x,t)$ by

$$v(x,t) = u(-x,t) \quad \text{for} \quad x \in (-a,a),\, t > 0.$$

Hence, if $u(x,t)$ and $v(x,t)$ are thus defined, then $v(x,t)$ satisfies

$$v_t = v_{xx} + v^p - v^\alpha |v_x|^{\beta-1} v_x \qquad \text{in} \quad (-a,a),\, t > 0,$$

$$v(x,0) = u(-x,0) = \varphi(-x), \qquad \text{in} \quad (-a,a),$$

$$v(\pm a,t) = 0, \qquad t > 0.$$

The results of Sections 2.2-2.4 can therefore be applied directly to the function $v(x,t)$ provided $\varphi(-x)$ satisfies the requirements previously made on $\varphi(x)$.

Hence, in the following analysis we need only consider the case when the gradient term is an even function of the gradient, i.e. $u(x,t)$ is taken as the solution to

$$u_t = u_{xx} + u^p + u^\alpha |u_x|^\beta \quad \text{in} \quad (-a,a), t > 0 \tag{2.5.1}$$

$$u(x,0) = \varphi(x) \quad \text{in} \quad (-a,a) \tag{2.5.2}$$

$$u(\pm a,t) = 0 \quad t > 0, \tag{2.5.3}$$

with

$$p > 1, \ \alpha,\beta \geq 0 \tag{2.5.5}$$

and φ satisfying

$$\varphi \in C^0[-a,a], \quad \varphi \geq 0, \quad \text{and} \quad \varphi(\pm a) = 0. \tag{2.5.6}$$

The following result is straightforward

Theorem 2.5.1

The solution, u , to the problem (2.5.1)-(2.5.3) will, if $\varphi(x)$ satisfies (2.5.5) and is 'large enough' compared to a , and α, β and p satisfy (2.5.4), blow-up in a finite time.

Proof

As the term $u^\alpha |u_x|^\beta$ in equation (2.5.1) is greater than or equal to zero we see immediately that

$$u_t \geq u_{xx} + u^p \quad \text{in} \quad (-a,a), t>0.$$

Hence, if $w(x,t)$ satisfies

$$w_t = w_{xx} + w^p \qquad \text{in} \quad (-b,b), t > 0,$$

$$w(x,0) = w_0(x) \qquad \text{in} \quad (-b,b),$$

$$w(\pm b) = 0 \qquad\qquad t > 0,$$

where w_0 is large enough to ensure that the results of Lacey (3) (Fujita (1969)) can be applied to this equation, then this function will blow-up in a finite time. Further, if $a \geq b$ and $\varphi(x) \geq w_0(x)$ for $x \in (-b,b)$ then w will be a lower solution to u which blows up in a finite time and hence Theorem 2.5.1.

Section 2.6 Identification of the blow-up sets

2.6.1 No blow-up for $x > 0$

Theorem 2.5.1 demonstrates that the solution to problem (2.5.1)-(2.5.4) will blow up in a finite time provided $\varphi(x)$ is sufficiently large. We now try to use the techniques developed in Sections 2.3 to identify when this blow-up takes place at a single point.

We proceed by assuming $\varphi(x)$ satisfies, in addition to (2.5.4),

$$\varphi \in C^1[-a, a], \tag{2.6.1}$$

and

$$\varphi'(x) \geq 0 \quad \text{if} \quad -a \leq x \leq x_0, \quad \varphi'(x) \leq 0 \quad \text{if} \quad x_0 \leq x \leq a, \tag{2.6.2}$$

for some $x_0 \in (-a, a)$.

In this analysis we shall also make the assumption that

$$\varphi(-x) \geq \varphi(x) \quad \text{for} \quad x \geq 0, \tag{2.6.3}$$

although the subsequent results would be equally valid had we chosen the opposite inequality (as in Section 2.3).

Hence, x_0 is necessarily less than or equal to zero.

Lemma 2.6.1

There exists a continuous function $s(t)$ with, if T is taken to be the finite blow up time of the function u, $-a < s(t) < a$ for $0 \le t < T$, such that

$$u_x(x,t) > 0 \text{ if } -a < x < s(t),\ 0 < t < T,$$

$$u_x(x,t) < 0 \text{ if } s(t) < x < a,\ 0 < t < T,$$

provided $\beta \ge 1$.

Proof

The maximum principle applied to (2.5.1)-(2.5.4) clearly establishes that any non-constant solution $u(x,t)$ is strictly positive throughout $(-a,a)x(0,T)$.

Next, if $u_x^\beta = |u_x|^\beta$, then on differentiating (2.5.1) we see that

$$u_{xt} = u_{xxx} + pu^{p-1}u_x + \alpha u^{\alpha-1}|u_x|^\beta u_x + \beta u^\alpha sign u_x |u_x|^{\beta-1}u_{xx},$$

to which, if $\beta \ge 1$, the maximum principle may be applied, in which case the remainder of this proof is identical to that of Lemma 2.3.1.

Lemma 2.6.2

If $u(x,t)$ is a solution to (2.5.1)-(2.5.3) to which Theorem 2.5.1 applies, and if $\varphi(x)$ satisfies (2.5.6) and (2.6.1)-(2.6.3), then

$$u(x,t) \le u(-x,t) \text{ for } x \ge 0 \qquad (2.6.4)$$

provided

$$p > 1, \quad \alpha \geq 0 \quad \text{and} \quad \beta \geq 1. \tag{2.6.5}$$

Proof

We consider the function

$$w(x,t) = v(x,t) - u(x,t) \quad \text{in} \quad R_\epsilon \tag{2.6.6}$$

where

$$v(x,t) = u(-x,t) \tag{2.6.7}$$

and

$$R_\epsilon = \{0 < x < a, 0 < t < T-\epsilon\} \quad \text{for all} \quad \epsilon > 0. \tag{2.6.8}$$

From (2.6.3) we have that

$$w(x,0) = \varphi(-x) - \varphi(x) \geq 0 \quad \text{for} \quad 0 \leq x \leq a. \tag{2.6.9}$$

Further,

$$w(0,t) = w(a,t) = 0 \quad \text{from (2.5.3)} \tag{2.6.10}$$

so that $w \geq 0$ on the parabolic boundary of R_ϵ

On differentiating (2.6.6) we see that

$$w_t = w_{xx} + v^p - u^p + v^\alpha \mid v_x \mid^\beta - u^\alpha \mid u_x \mid^\beta$$

i.e. $$w_t = w_{xx} + cw + d \mid v_x \mid^\beta w + e v^\alpha w_x, \tag{2.6.11}$$

where

$$c = \frac{(u^p - v^p)}{(u-v)}, \quad d = \frac{(u^\alpha - v^\alpha)}{(u-v)}, \quad \text{and} \quad e = \frac{(\mid u_x \mid^\beta - \mid v_x \mid^\beta)}{(u_x - v_x)}.$$

The maximum principle applied to (2.6.11) yields that a negative interior minimum of w will be impossible except perhaps at some point where one of the functions c, d or e is unbounded.

However, $p > 1$ and $\beta \geq 1$ (from (2.6.6)) and both terms c and e must remain bounded throughout the interior of R_ϵ.

Further, if $0 \leq \alpha < 1$, then d may become unbounded at some point for which $u = v$. At such a point, however, we would have that $w = v - u = 0$ which would contradict this point being a negative minimum of w.

It follows, therefore, as w is no less than zero on the parabolic boundary of R_ϵ, that $w \geq 0$ throughout R_ϵ and hence Lemma 2.6.2

Lemmas 2.6.1 and 2.6.2 yield

Corollarly 2.6.3

$$s(t) \leq 0 \quad \text{and} \quad u_x(x,t) < 0 \quad \text{for} \quad 0 < x < a, \quad 0 < t < T.$$

If s_+ and s_- are again defined as

$$s_- = \liminf_{t \to T} s(t) \quad \text{and} \quad s_+ = \limsup_{t \to T} s(t) \quad , \tag{2.6.12}$$

then by Corollary 2.6.3, $s_+ \leq 0$.

Theorem 2.6.4

For any $\epsilon_0 > 0$ there exists a constant $C > 0$ such that

$$u(x,t) \leq C \quad \text{for} \quad s_{+}+\epsilon_0 \leq x \leq a, \; 0 < t < T$$

if $\beta > 1$ and $p > \alpha+\beta$, or if $\beta = 1$ and $p > 2\alpha+1$, and there is no blow-up is $x > 0$.

Proof

The proof of this result is identical to the proof of Lemma 2.3.5 in the case $u_x^\beta = |u_x|^{\beta-1}u_x$ and is not repeated.

2.6.2 Single point blow-up

In this section we continue the study of solutions to the problem (2.5.1)-(2.5.3) for which finite time blow-up does occur by extending the conclusions of Theorem 2.6.4 to the left of $\{ x = s(t) \}$.

Lemma 2.6.5

For any $\epsilon_0 > 0$ there exists a constant $C > 0$ such that

$$u(x,t) \le C \quad \text{for} \quad s_- - \epsilon_0 \ge x \ge -a, 0 < t < T$$

if $\beta > 1$ and $p > \alpha + \beta$, or if $\beta = 1$ and $p > 2\alpha + 1$.

Proof

The proof of this result is also very similar to that of Lemma 2.3.5.

In this case, the function $J(x,t)$ is defined as

$$J(x,t) = u_x(x,t) - \epsilon(\delta - x)^n u^k \quad \text{in} \quad R_1 \tag{2.6.13}$$

for

$$R_1 = \{-a \le x \le \delta, T_0 \le t \le T\} \tag{2.6.14}$$

and where $\delta = s_- - \tfrac{1}{2}\epsilon_0$ for any $\epsilon_0 > 0$. (2.6.15)

In addition, $k > 1$, $n > 1$, ϵ is assumed small and T_0 is close enough to T to ensure that

$$s(t) > \delta \quad \text{for} \quad T_0 \le t < T. \tag{2.6.16}$$

On differentiating (2.6.13) and following suitable manipulation it can be established that

$$J_t - J_{xx} - DJ - EJ_x \geq S_1 \tag{2.6.17}$$

for functions D and E which must remain bounded at any interior minimum of the function J and where

$$\begin{aligned} S_1 = \epsilon(\delta-x)^n \{ & (p-k)u^{p+k-1} - n\beta\epsilon^{\beta-1}(\delta-x)^{n(\beta-1)-1}u^{\alpha+k\beta} \\ & + \frac{n(n-1)u^k}{(\delta-x)^2} + 2n\epsilon(\delta-x)^{n-1}u^{2k-1} \\ & + \epsilon^\beta(\delta-x)^{n\beta}(\alpha+k(\beta-1))u^{\alpha-1+k(\beta+1)}\}. \end{aligned} \tag{2.6.18}$$

The sign of the term S_1 can also be determined by the arguments used in Section 2.3.2 to identify the sign of the right hand side of equation (2.3.60). The proof of Lemma 2.3.5 shows that the right hand side of equation (2.3.60) is no greater than zero throughout the considered region provided $\alpha \geq 0$, $p \geq 1$ and if

$$\beta > 1 \text{ and } p > \alpha + \beta, \text{ or if } \beta = 1 \text{ and } p > 2\alpha + 1, \tag{2.6.19}$$

for ϵ and $1/n$ sufficiently small.

These arguments are equally valid in the present context, however, and allow the conclusion that S_1 , as defined by (2.6.19), is no <u>less</u> than zero throughout R_1 provided (2.6.19) holds.

Further,

$$J(-a,t) = u_x(-a,t) - \epsilon(\delta+a)^n u^k(-a,t) \qquad \text{by Lemma 2.6.2 and (2.5.3).}$$
$$= u_x(-a,t) > 0$$

$$J(\delta,t) = u_x(\delta,t) - \epsilon(\delta-\delta)^n u^k(\delta,t) \qquad \text{for } T_0 \le t < T \text{ by (2.6.16)}$$
$$= u_x(\delta,t) > 0$$

and

$$J(x,T_0) = u_x(x,T_0) - \epsilon(\delta-x)^n u^k(x,T_0) \qquad \text{by Lemma 2.6.2. and (2.6.16)}$$
$$> 0$$

provided ϵ is small enough.

Hence, $J > 0$ on the parabolic boundary of R_1 and the maximum principle yields

$$J(x,t) = u_x(x,t) - \epsilon(\delta-x)^n u^k(x,t) > 0$$
$$\text{for } -a \le x \le \delta,\ T_0 \le t < T\,, \tag{2.6.20}$$

if ϵ and $\frac{1}{n}$ are sufficiently small and condition (2.6.19) holds.

Finally, (as in Theorem 2.3.4), we may integrate inequality (2.6.20) from δ to x for any $-a \le x < \delta$ to obtain

$$u(x,t) \le \{\,\epsilon(k-1)(\delta-x)^{n+1}/(n+1)\,\}^{\frac{-1}{k-1}} .$$

This bound on u is increasing in x and we consequently see that, if

$\delta_1 - \delta - \frac{1}{2}\epsilon_0 - s_- - \epsilon_0$, then

$$u(x,t) \le \{e(k-1)(\delta-\delta_1)^{n+1}/(n+1)\}^{-1/(k-1)} \\ - C \tag{2.6.21}$$

for any $-a \le x \le s_- - \epsilon_0 - \delta_1,\ T_0 \le t \le T$.

Inequality (2.6.21) is clearly also automatic for $0 \le t < T_0$ and hence

Lemma 2.6.5.

Theorem 2.6.4 and Lemma 2.6.5 combine to yield the following corollary.

Corollary 2.6.6

If in addition to (2.6.5) the parameters $\alpha,\ \beta$ and p satisfy one of the relations described by (2.6.19), then a solution, $u(x,t)$, to problem (2.5.1)-(2.5.3) having $\varphi(x)$ satisfying (2.5.5) and (2.6.1)-(2.6.3), is bounded within the regions

$$\{-a \le x \le s_- - \epsilon_0, \quad 0 \le t < T\} \cup \{s_+ + \epsilon_0 \le x \le a, \quad 0 \le t < T\}$$

for any $\epsilon_0 > 0$.

Theorem 2.6.7

If $u(x,t)$ is a solution to (2.5.1)-(2.5.3) which satisfies Theorem 2.5.1 (so that u blows up at some time $T < \infty$) and if $\varphi(x)$ satisfies conditions (2.5.5) and (2.6.1)-(2.6.3), and

$$\varphi'' + \varphi^p + \varphi^\alpha |\varphi'|^\beta \geq 0 \quad \text{in} \quad (-a,a) \ , \tag{2.6.22}$$

then blow-up will occur at a single point provided the conditions (2.6.5) and (2.6.19) hold.

Proof

In light of Corollary 2.6.6 it suffices to show that $s_+ = s_-$.

The condition (2.6.22) shows that $u_t \geq 0$ at $t = 0$. On differentiating (2.5.1) with respect to t and applying the maximum principle (as in Section 2.3.2) we are able to conclude that

$$u_t > 0 \quad \text{for} \quad -a < x < a, \ 0 < t < T \ . \tag{2.6.23}$$

We next assume that $s_- < s_+$ and set

$$s_1 = s_- + \epsilon, \ s_2 = s_+ + \epsilon \quad \text{and} \quad \theta = \tfrac{1}{2}(s_1 + s_2) \ , \tag{2.6.24}$$

where $3\epsilon < s_+ - s_-$.

We consider a region

$$Q = \{ s_1 < x < s_2, \ T_0 \leq t < T \} \tag{2.6.25}$$

where T_0 is close enough to T so that

$$s(T_0) < s_1 \tag{2.6.26}$$

and

$$u(s_1,t) > u(s_2,t) \quad \text{for} \quad T_0 \le t < T . \tag{2.6.27}$$

Hence we use the facts that

$$u(s_1,t) \to \infty \quad \text{as} \quad t \to T$$

and $\quad u(s_2,t) \le c < \infty \quad$ for $\quad 0 < t < T$

from Theorem 2.6.5, (2.6.23) and (2.6.24).

Introduce the function

$$w(x,t) = u(x,t) - v(x,t) \tag{2.6.28}$$

where

$$v(x,t) = u(2\theta - x, t) \tag{2.6.29}$$

and we try to establish that $w > 0$ in $Q \cap \{x < \theta\}$. From (2.6.28), (2.6.26), and (2.6.27) we see that

$$w(\theta,t) = u(\theta,t) - u(\theta,t) = 0 ,$$

$$w(s_1,t) = u(s_1,t) - u(s_2,t) > 0 \quad \text{for} \quad T_0 \le t < T$$

and $\quad w(x,T_0) = u(x,T_0) - v(x,T_0) > 0$

if T_0 is close enough to T .

Hence, $w \ge 0$ on the parabolic boundary of $Q \cap \{x < \theta\}$. We now proceed, as in Lemma 2.6.2, to show that $w > 0$ in $Q \cap \{x < \theta\}$.

On differentiating (2.6.28) with respect to t, we find that w satisfies

$$
\begin{aligned}
w_t &= w_{xx} + u^p - v^p + u^\alpha |u_x|^\beta - v^\alpha |v_x|^\beta \\
&= w_{xx} + u^p - v^p + \{u^\alpha - v^\alpha\} |u_x|^\beta + u^\alpha\{|u_x|^\beta - |v_x|^\beta\} \\
&= w_{xx} + cw + dw|u_x|^\beta + eu^\alpha w_x,
\end{aligned}
\tag{2.6.30}
$$

where

$$c = \frac{(u^p - v^p)}{u-v}, \quad d = \frac{(u^\alpha - v^\alpha)}{u-v}, \quad \text{and} \quad e = \frac{|u_x|^\beta - |v_x|^\beta}{u_x - v_x}.$$

The maximum principle applied to (2.6.30) establishes that a negative minimum of w is impossible within the interior of $\varrho \cap \{x < \theta\}$, except possibly at some point where one of the functions c, d or e is unbounded. As $p > 1$ and $\beta \geq 1$ from (2.6.5), however, each of the functions c and e must remain bounded throughout the considered region. Further, d can only become unbounded, if $0 \leq \alpha < 1$, at some point where $u = v$. It is clear, therefore, that a negative minimum of w is also impossible at such a point.

We conclude, therefore, from the maximum principle applied to (2.6.30), that $w > 0$ within $\varrho \cap \{x < \theta\}$, and hence that

$$s(t) < \theta \quad \text{for} \quad T_0 \leq t < T. \tag{2.6.31}$$

From (2.6.31), it follows that

$$s_+ = \limsup_{t \to T} s(t) \leq \theta = \tfrac{1}{2}(s_+ + s_-) + \varepsilon$$

so that $s_+ \leq s_- + 2\epsilon$

which contradicts the definition of ϵ .

Hence, $s_+ = s_-$, and blow-up occurs at the single point

$$s_+ = s_- = \lim_{t \to T} s(t) \quad .$$

Section 2.7 Estimate of blow-up rate for the positive gradient case

In this section we find that the techniques of Section 2.4 can be applied directly to equations of the form of (2.5.1)-(2.5.3) and yield valuable information on the rate of single-point blow-up.

We consider a solution, $u(x,t)$, to problem (2.5.1) -(2.5.3) to which

Theorem 2.5.1 applies (so that u blows up at the finite time $T < \infty$).

Theorem 2.4.1 of Section 2.4 can be applied without adjustment to this function, and along with its corollary, Corollary 2.4.2, yields:-

Theorem 2.7.1

If the function $m(t)$ is defined as

$$m(t) = \max_{x \in (-a,a)} u(x,t) \ , \tag{2.7.1}$$

then $m(t)$ is Lipschitz continuous, and

$$m'(t) \le m^p(t) \tag{2.7.2}$$

at any point at which m is differentiable.

On integrating (2.7.2) we find that there exists a constant $c > 0$ such that

$$m(t) \ge \frac{c}{(T-t)^{1/(p-1)}} \text{ , for any } 0 < t < T \ . \tag{2.7.3}$$

We next assume (as in Theorem 2.6.7), that the initial condition $\varphi(x)$ satisfies (2.5.5) and (2.6.1)-(2.6.3) and (2.6.22). This again allows the conclusion that

$$u_t > 0 \quad \text{for} \quad -a \le x \le a, \quad 0 < t < T. \tag{2.7.4}$$

Proceeding as in Section 2.4, we consider a region, Ω^η, defined as

$$\Omega^\eta = \{-a+\eta < x < a-\eta\} \quad \text{for any} \quad \eta > 0, \tag{2.7.5}$$

and the function $J(x,t)$

$$J(x,t) = u_t(x,t) - \epsilon e^{-Mt} u^s(x,t) \tag{2.7.6}$$

where $\epsilon, s > 0$ with ϵ small, and M is a positive constant. We wish to show that $J > 0$ within Ω^η.

If the conditions (2.6.5) and (2.6.19) hold, then Theorem 2.6.7 applies and u will blow-up at a single interior point of Ω_T. Hence, if η is small enough, it follows that

$$u^s \le C_0 < \infty \quad \text{for} \quad x = \pm(a-\eta), \quad 0 < t < T. \tag{2.7.7}$$

We also have from (2.7.4) that

$$u_t \ge c > 0 \tag{2.7.8}$$

on the parabolic boundary of the set $\Omega^\eta x(\eta, T)$.

If ϵ is sufficiently small, then (2.7.7) and (2.7.8) ensure that $J(x,t)$ can be chosen to be strictly positive on the parabolic boundary of $\Omega^\eta x(\eta, T)$.

On differentiating (2.7.7) it can be established (as in Section 2.4), that

$$J_t - J_{xx} - \beta u^{\alpha} \, | u_x |^{\beta-1} sign(u_x) J_x \\ - \{p u^{p-1} + \alpha u^{\alpha-1} \, | u_x |^{\beta}\} J = S, \tag{2.7.9}$$

where

$$S = \epsilon e^{-\mathbf{M}t}\{(p-s) u^{p+s-1} + (\alpha+s(\beta-1)) u^{\alpha+s-1} \, | u_x |^{\beta} + \mathrm{M} u^{s} + s(s-1) u^{s-2} u_x^2\}. \tag{2.7.10}$$

Further, because $u > 0$ throughout $\Omega^{\eta}x(\eta, T)$, and as $\beta \geq 1$ from (2.6.5), it is clear that the coefficients of J and J_x in equation (2.7.9) are bounded functions within $\Omega^{\eta}x(\eta, T)$.

The condition (2.6.5) ensures that $\beta \geq 1$, and hence that S will be greater than or equal to zero throughout $\Omega^{\eta}x(\eta, T)$, if

$$p \geq s \geq 1. \tag{2.7.11}$$

The maximum principle may be applied to (2.7.9) to yield that $J(x, t)$, as defined by (2.7.6), is strictly greater than zero throughout $\Omega^{\eta}x(\eta, T)$, and hence that

$$u_t(x, t) \geq \epsilon e^{-\mathbf{M}t} u^{s}(x, t) \quad \text{in} \quad \Omega^{\eta}x(\eta, T). \tag{2.7.12}$$

As $u > 0$ within $\Omega^{\eta}x(\eta, T)$, we may integrate (2.7.12) from t to T and find that there exists a constant $C > 0$ such that

$$u(x,t) \leq \frac{C}{(T-t)^{1/(s-1)}} \quad \text{for} \quad (x,t) \in \Omega^{\eta} x (\eta, T),$$

provided η is small enough and s satisfies (2.7.11).

Further, if η is small enough, then it follows from Theorem 2.6.7 that u must also remain bounded at any other point within Ω_T.

Hence, as $s - p$ satisfies (2.7.11) we find that

$$u(x,t) \leq \frac{C}{(T-t)^{1/(p-1)}} \quad \text{for} \quad (x,t) \quad \text{in} \quad \Omega_T, \tag{2.7.13}$$

and we have established the following Theorem:-

Theorem 2.7.2

If $u(x,t)$ is a solution to (2.5.1)-(2.5.3) which satisfies Theorem 2.5.1, and if $\varphi(x)$ satisfies (2.5.6), (2.6.1)-(2.6.3) and (2.6.22), then there exists a constant $C > 0$ for which inequality (2.7.13) holds provided α, β and p satisfy (2.6.5) and (2.6.19).

Chapter 3 The Higher Dimensional Gradient Problem

3.1 Introduction

Chapter 2 has considered the question of when solutions to the problem (2.1.1)-(2.1.3) in one dimension will blow-up in a finite-time and when this blow-up can be identified as occurring at a single point. In this chapter we shall consider the problem(s) (2.1.1)-(2.1.3) in higher dimensions with the aim of developing extensions of the one-dimensional results established in Chapter 2.

As in Chapter 2, we find it convenient to treat the cases of positive and negative gradient terms in (2.1.1) as entirely separate problems which are studied in the same order. Hence the case of a negative gradient term in the N-dimensional problem (2.1.1)-(2.1.3) is investigated in Sections 3.2-3.4 and the positive gradient problem in Sections 3.5-3.7. The various 'blow-up questions' addressed within these subsections also correlate with those considered in Chapter 2 and this should facilitate necessary cross-referencing.

By using the techniques developed in Friedman & Lacey 1988 as a starting point, Chapter 2 has been able to construct a method whereby useful information can be obtained about the blow-up behaviour of the one-dimensional problem with both positive and negative gradient terms.

When the gradient term in the one-dimensional form of equation (2.1.1) is of negative sign, this process relies on the development, in Appendix A, of a 'good' upper bound for the gradient of the considered solution. The extension of the methods used in Sections 2.2-2.4 to higher dimensions is consequently made possible because this upper bound also applies to solutions to the N-dimensional problem. (Provided the boundary of the domain, Ω , is taken to have non-negative mean curvature.)

The technique applied in Section 2.2 to establish blow-up in the 'negative gradient' case is also relatively 'self-contained'. It essentially requires only that a number of estimates, derived in Friedman & MacLeod 1985, be valid for the corresponding equation without the gradient term, and that this equation has solutions which exhibit finite-time blow-up.

As none of these results are particular to the one-dimensional problem, we begin the study of the N-dimensional problem (2.1.1)-(2.1.3) in which the gradient term is of negative sign by extending the techniques of Section 2.2 into higher dimensions. This will allow an N-dimensional analogue of Theorem 2.2.3 to be established.

It is unlikely that the analysis of Section 2.3 (in which single-point blow-up is established for the one-dimensional negative gradient problem) will be able to identify single-point blow-up for anything other than a radially symmetric solution to the higher dimensional problem.

This opinion does not stem from any preconceived ideas of when blow-up may occur at a single-point in the N-dimensional case, but is simply a recognition of the higher degree of dependence placed by the techniques used on the existence of only one important space variable.

Hence it is anticipated that information about the blow-up sets for the non-symmetric N-dimensional problem may be best facilitated by extensions, where practical, of techniques used in Friedman & MacLeod 1985. In this work it is found that, for higher dimensional problems with suitable initial data, blow-up will occur (in the absence of the gradient term) within a compact subset of a convex domain.

In Section 3.3 we begin our analysis of where blow-up may occur by extending the techniques used to establish single-point blow-up in Section 2.3 to radially symmetric solutions of the N-dimensional problem with a negative gradient term (Section 3.3.1). The possible extension of techniques similar to those used by Friedman & MacLeod to non-symmetric solutions of this problem is considered in Section 3.3.2.

In Section 3.4 we address the problem of deriving an estimate of the rate at which finite-time blow-up may occur, and find that such an estimate is available provided the results of the preceding subsections hold.

The case of a positive gradient term in (2.1.1) is considered in Sections 3.5-3.7. Again this analysis is strongly influenced by the study of the corresponding one-dimensional problem (in Sections 2.5-2.7) and here we are also able to draw on experience gained in the higher dimensional analysis of Sections 3.2-3.4. Hence in Section 3.5 we investigate the existence of blow-up when the gradient term in equation (2.1.1) is of positive sign, Section 3.6 considers the problems associated with identifying the blow-up sets, and Section 3.7 seeks to establish an estimate of the rate at which finite-time blow-up may occur.

Section 3.2 Existence of blow-up for a negative gradient term

3.2.1 Blow-up using the techniques of Friedman and Lacey

In this section we consider the N-dimensional problem (2.1.1)-(2.1.3) in which the gradient term is of negative sign.

Hence, $u(x,t)$ is taken to be the solution to

$$u_t = \nabla^2 u + u^p - u^\alpha |\nabla u|^\beta \quad \text{in} \quad \Omega,\ t > 0 \tag{3.2.1}$$

$$u(x,0) = \varphi(x) \qquad \text{in} \quad \Omega \tag{3.2.2}$$

$$v(x,t) = 0 \qquad \text{on} \quad \partial\Omega,\ t > 0 \tag{3.2.3}$$

where Ω is a bounded region in $\mathbf{R}^N$ with smooth boundary $\partial\Omega$. The parameters $\alpha,\ \beta$ and p are assumed to satisfy the condition

$$p > 1,\ \alpha \geq 0 \ \text{ and } \ \beta \geq 1 \ . \tag{3.2.4}$$

In addition, we shall assume throughout this work that the initial condition $\varphi(x)$ satisfies the requirement introduced in Section 2.2 as (2.2.5) and restated as

$$\varphi \in C^0(\Omega),\ \varphi \geq 0 \ \text{ and } \ \varphi(x) = 0 \ \text{ for } x \text{ on } \partial\Omega \ . \tag{3.2.5}$$

Standard iteration methods can be used to show that the problem (3.2.1)-(3.2.3) along with (3.2.4) and (3.2.5) will have a unique, positive solution, $u(x,t)$, at least until some small time T_0 . Further, if u cannot be extended step-by-step to all t, then there exists a finite time T such that u exists and is positive for $0 < t < T$, and

$$\limsup_{t \to T} \sup_{x \in \Omega} u(x,t) = +\infty,$$

and hence that u blows up at the finite time T.

We begin this study of the blow-up behaviour of solutions to (3.2.1)-(3.2.4) by developing an analogue of Theorem 2.2.1 for this higher-dimensional problem. This technique consequently represents the direct application of the methods used in Friedman & Lacey 1988 in an attempt to establish the existence of blow-up in the N-dimensional case.

This process proceeds by first considering a function $v(x,t)$ which is assumed to satisfy

$$v_t = \nabla^2 v + v^p \qquad \text{in } B,\ t > 0, \tag{3.2.6}$$

$$v(x,0) = \psi(x) \qquad \text{in } B, \tag{3.2.7}$$

$$v(x,t) = 0 \qquad \text{on } \partial B,\ t > 0, \tag{3.2.8}$$

for B, a ball, defined as

$$B = \{ x : |x| < b \} \text{ for some } b > 0. \tag{3.2.9}$$

Additionally, the initial condition $\psi(x)$ is taken to satisfy

$$\psi = \psi(r) \in C^1(B),\ \psi'(r) < 0, \text{ if } 0 < r < b \text{ and } \psi''(r) < 0. \tag{3.2.10}$$

It follows that if $\psi(r)$ is 'large enough' compared to b then v will blow up at a finite time, say t_0 (Fujita 1969, Bebernes & Kassoy 1981, Lacey 1983). Further, in light of (3.2.10), the maximum principle applied to v_r yields that

$$v = v(r,t) \quad \text{and} \quad v_r < 0 \quad \text{for} \quad 0 < v < b, \quad 0 < t < t_0 \,. \tag{3.2.11}$$

The following estimates, derived in Friedman & MacLeod 1985 as Theorem 4.2 and Theorem 3.1 respectively, were used, in their one-dimensional form, in Section 2.2. They are, however, equally valid when applied to the function v whereby (as B is necessarily convex) we see that

$$v(r,t) \le C(t_0-t)^{-1/(p-1)} \quad \text{for} \quad 0 \le r < b \,, \; 0 \le t < t_0 \,, \tag{3.2.12}$$

for some positive constant C , and

$$\tfrac{1}{2}|v_r|^2 = \tfrac{1}{2}|\nabla v|^2 \le \int_{v(r,t)}^{m(t)} s^p \, ds \quad \text{for} \qquad 0 \le r < b \,, \; 0 \le t < t_0 \,. \tag{3.2.13}$$

In light of (3.2.11), we also have that the function, $m(t)$, appearing in (3.2.13) satisfies

$$m(t) = \max_{x \in B} v(x,t) = v(0,t) \,, \qquad \text{for} \quad 0 < t < t_0 \,. \tag{3.2.14}$$

Hence, if the function $r(t)$ is defined as

$$r(t) = \int_0^t m^\gamma(\tau)\, d\tau, \quad r_0 = r(t_0) \,, \tag{3.2.15}$$

then by (3.2.12),

$$r_0 < \infty \text{ provided } \gamma < p-1 \ . \tag{3.2.16}$$

We next define the regions R_1 and R_2 as

$$R_1 = \{0 \le r < \delta(t),\ 0 < t < t_0\} \tag{3.2.17}$$

$$R_2 = \{\delta(t) < r < b + \delta(t),\ 0 < t < t_0\} \tag{3.2.18}$$

and a function, $w(r,t)$, by

$$w(r,t) = v(0,t) = m(t) \quad \text{in } R_1 \tag{3.2.19}$$

$$w(r,t) = v(r - \delta(t), t) \quad \text{in } R_2 \ , \tag{3.2.20}$$

where

$$\delta(t) = r(t_0) - r(t) \ge 0 \ , \tag{3.2.21}$$

and we propose that, under appropriate conditions on the relationship between the parameters α, β and p , and provided $\varphi(x)$ is chosen suitably, $w(r,t)$ will be a subsolution to $u(x,t)$, the solution to (3.2.1)-(3.2.4) and (3.2.5).

To verify this proposition we proceed as follows:

In R_1 ,

$$\begin{aligned} w_t - \nabla^2 w - w^p + w^\alpha |\nabla w|^\beta &= m'(t) - m^p(t) \\ &\le (v_t - \nabla^2 v + v^\alpha |\nabla v|^\beta)(0,t) - v^p(0,t) \\ &\le 0 \end{aligned}$$

as $\nabla^2 v \leq 0$ and $|\nabla v| = 0$ at $r = 0$.

Next, in R_2 ,

$$
\begin{aligned}
& w_t - \nabla^2 w - w^p + w^\alpha |\nabla w|^\beta \\
& = w_t - w_{rr} - \frac{(N-1)}{r} w_r - w^p + w^\alpha \,|\, w_r \,|^\beta \\
& = v_{rr} + \frac{(N-1)}{(r-\delta)} v_r + v^p - \delta' v_r - v_{rr} - \frac{(N-1)}{r} v_r - v^p + v^\alpha \,|\, v_r \,|^\beta \\
& = (N-1)\, v_r \left[\frac{1}{r-\delta} - \frac{1}{r} \right] - \delta' v_r + v^\alpha \,|\, v_r \,|^\beta .
\end{aligned}
\tag{3.2.22}
$$

Hence, as $v_r < 0$ in R_2 (from (3.2.21)) and as $\frac{1}{r-\delta} - \frac{1}{r} \geq 0$ in R_2 , we see

that the right hand side of (3.2.22) is less than or equal to

$$
|v_r| \left\{ v^\alpha |v_r|^{\beta-1} + \delta'(t) \right\}. \tag{3.2.23}
$$

Differentiating (3.2.21), however, we see that

$$
\delta'(t) = -m^\gamma(t) \ ,
$$

so that (3.2.13) becomes

$$
|v_r| \left\{ v^\alpha |v_r|^{\beta-1} - m^\gamma(t) \right\}. \tag{3.2.24}
$$

Hence, if (3.2.13) is used to estimate $|v_r|$ in (3.2.24), it follows, as

$v(r,t) \le m(t)$ for $0 \le r < b$, and from (3.2.24), as $\alpha \ge 0$ and $\beta \ge 1$, that

(3.2.24) is itself no greater than

$$|v_r| \{ c_1 m^{\alpha + \frac{1}{2}(p+1)(\beta-1)} - m^{\gamma} \} \tag{3.2.25}$$

for some constant c_1 .

Hence, the right hand side of (3.2.22) must be less than or equal to zero if the expression (3.2.25) is less than or equal to zero, which it will be if

$$c_1 m^{\alpha + \frac{1}{2}(p+1)(\beta-1)} \le m^{\gamma}. \tag{3.2.26}$$

It follows therefore, that if $m(t)$ is sufficiently large, inequality (3.2.26) will be satisfied as required if

$$\alpha + \tfrac{1}{2}(p+1)(\beta-1) < \gamma < p-1 . \tag{3.2.27}$$

By definition, however, $m(t) = \max_{x \in B} v(x,t)$, and the maximum principle applied

to (3.2.6)-(3.2.8) yields that

$$m(t) \ge v(r,t) \ge v(r,0) = \psi(r) \qquad \text{for } 0 \le r < b,\ 0 < t < t_0$$

so that $m(t)$ can be guaranteed as large as required by suitable choice of the

function $\psi(r)$.

Hence, as both w and ∇w are continuous across the boundary $\partial R_1 \cap \partial R_2$, with $\nabla w = 0$ on this curve, we conclude that w satisfies the requirements necessary to be a subsolution to u in the interior, B' , of the set $\bar{R}_1 \cup \bar{R}_2$. Further, w is continuous on the parabolic boundary $\partial_p B'$, of B' , with

$$w(r,0) = \psi(r) \qquad \text{for} \quad 0 \le r < b'$$

and

$$w = 0 \qquad\qquad \text{elsewhere on} \quad \partial_p B' \ ,$$

where $b' = b + \delta(0) = b + r_0$.

It follows, therefore, that if $B' \subset \Omega$, and if $\varphi(x) > \psi(x)$ for $x \in B'$, then by comparison

$$u(x,t) \ge w(x,t) \ . \tag{3.2.28}$$

In conclusion, the results of Lacey 1983 may be applied to the function $v(r,t)$, to show that, if $\psi(r)$ is 'large enough' compared to b , and consequently if $\varphi(x)$ is also 'large enough' compared to Ω , then $v(r,t)$ exhibits finite-time blow-up, and hence that $w(r,t)$ is a subsolution to u which blows up in a finite time.

We have therefore established the following N-dimensional extension of the results of Theorem 2.2.1:-

Theorem 3.2.1

If $\varphi(x)$ is 'large enough' compared to Ω (so that inequality (3.2.28) is valid for a function $w(x,t)$ which exhibits finite time blow-up) and if the parameters $\alpha,\ \beta$ and p satisfy, in addition to (3.2.4), the relationship described in (3.2.27), then $u(x,t)$, the solution to (3.2.1)-(3.2.4) with $\varphi(x)$ satisfying (3.2.5) will blow up in a finite time less than or equal to t_0 .

Remark 3.2.2

It is also a straightforward process to verify that no solution to (3.2.1)-(3.2.4) can exhibit finite time blow-up if $\alpha + \beta \geq p$.

Proof

Bearing in mind Remark 2.2.2 of Section 2.2, we define a function, $W(x,t)$, as

$$W(x,t) = Ae^{b(x_1+a)} \tag{3.2.29}$$

for positive constants $A,\ b$ and a , and for any space variable x_1 . We also assume that a is large enough to ensure that

$$x_1 + a \geq 0 \qquad \text{for any } x\in\Omega \ . \tag{3.2.30}$$

This function $W(x,t)$ therefore satisfies

$$\begin{aligned} &W_t - \nabla^2 W - W^p + W^\alpha |\nabla W|^\beta \\ &\quad = A^p e^{pb(x_1+a)}\{ A^{(\alpha+\beta-p)} b^\beta e^{(\alpha+\beta-p)b(x_1+a)} - 1 \} - Ab^2 e^{b(x_1+a)} \end{aligned} \tag{3.2.31}$$

and we see that, if we substitute x for x_1 in (3.2.31) we obtain an identical expression to that found in the proof of Remark 2.2.2 in Section 2.2 i.e. (2.2.23). An identical argument can therefore be used to establish that the right hand side of (3.2.31) will remain greater than or equal to zero for all x in Ω and $t \geq 0$ provided $A,\ b > 1$, with A 'large enough' compared to b , and

$$\alpha + \beta \geq p > 1. \tag{3.2.32}$$

Further, (3.2.30) allows us to estimate

$$W(x,t) \geq A > 0 \ , \quad \text{for both} \quad t = 0 \quad \text{and} \quad x \in \Omega \ ,$$

$$\text{or for} \quad x \in \partial\Omega \quad \text{and} \quad t > 0 \ .$$

Hence, a suitably large choice of constant A ensures, from the maximum principle, that $W(x,t)$ is a supersolution to any solution u of (3.2.1)-(3.2.4), which must consequently remain finite for all times t provided condition (3.2.32) holds. Hence Remark 3.2.2.

3.2.2 A 'stronger' blow-up result

Section 3.2.1 has found that the results of Section 2.2.1 can be established, without alteration, for the general N-dimensional problem (3.2.1)-(3.2.4). We therefore once again consider the question addressed at the beginning of Section 2.2.2; can a solution to (3.2.1)-(3.2.4) exhibit, for suitable initial data, finite time blow-up for values of the parameters α, β and p outwith the scope of either

Theorem 3.2.1 or Remark 3.2.2? i.e. for $\alpha \geq 0$, $\beta \geq 1$ and $p > 1$, such that

$$\frac{3p - 2\alpha - 1}{p + 1} \leq \beta < p - \alpha . \tag{3.2.33}$$

The aim of this section is simply to try to answer this question, by showing that the techniques developed in Section 2.2.2 may also be extended to the higher dimensional problem and that an N-dimensional analogue of Theorem 2.2.3 is valid for the solutions to (3.2.1)-(3.2.4).

We begin by considering the solution $z(x,t)$ to the following problem:-

$$z_t = \nabla^2 z + z^p - z^q |\nabla z| \quad \text{in } B,\ t>0 \tag{3.2.34}$$

$$z(x,0) = \psi(x) \quad \text{in } B \tag{3.2.35}$$

$$z(x,t) = 0 \quad \text{for } x \text{ on } \partial B,\ t>0 \tag{3.2.36}$$

where B is again defined as a ball such that

$$B = \{ x : |x| < b \} \tag{3.2.27}$$

and where $\psi(x)$ is assumed to satisfy

$$\psi \in C^0(B),\ \psi \geq 0 \quad \text{and} \quad \psi(x) = 0 \quad \text{for} \quad x \quad \text{on} \quad \partial B,\ t > 0. \tag{3.2.28}$$

The combined results of Theorem 3.2.1 and Remark 3.2.2 may be applied to this function and yield that, if ψ is 'large enough' compared to B , then $z(x,t)$ will blow-up in a finite time, say T , if

$$0 \leq q < p - 1 \tag{3.2.39}$$

from Theorem 3.2.1. Further, $z(x,t)$ will remain finite for all times t , and will consequently not exhibit finite-time blow-up, if

$$q \geq p - 1 > 0 \tag{3.2.40}$$

from Remark 3.2.2.

It is clear, therefore, that the problem addressed in this section can be answered in the case $\beta = 1$ by a straightforward application of the results of Section 3.2.1.

We proceed as in Section 2.2.2 by seeking to duplicate the arguments used in Section 3.2.1, where in this case we will use the function $z(x,t)$ in place of the function $v(x,t)$ (the solution to (3.2.6)-(3.2.9)) used previously.

This is only possible if an upper bound for the gradient of this solution exists, and such a bound has been established in Appendix A.

Appendix A shows that, if $u(x,t)$ is the solution to problem (3.2.1)-(3.2.4) in a convex domain Ω , then

$$|\nabla u|^2 \le \{ Cu^m + u^k - Au^l + B \}^2$$

for positive constants A, B and C, with B large compared to A and C, provided

(i) $\beta > \dfrac{2(p-\alpha)}{(p+1)}$,

(ii) $C^\beta [\alpha + n(\alpha + m(\beta-1))] > p$,

(iii) $C \ge 1$ and $m - (p-\alpha)/\beta > k > l > 1$,

where k is chosen close enough to m and n is the integer part of β, i.e. the largest integer satisfying $n \le \beta$.

This estimate may be applied to $z(x,t)$, the solution (3.2.34)-(3.2.38), and yields that

$$|\nabla z|^2 \le \{ Cz^m + z^k - Az^l + B \}^2 \tag{3.2.41}$$

provided

(i) $q > \frac{1}{2}(p-1)$,

(ii) $q > \dfrac{p}{2C}$,

(iii) $C \ge 1$ and $m - (p-q) > k > l > 1$,

for k close enough to $m - (p-q)$.

If $z(x,t)$ is taken to be a solution to (3.2.34)-(3.2.36) to which Theorem 3.2.1 applies, i.e. if $\psi(x)$ is large enough compared to the ball B and if condition (3.2.39) is satisfied, we denote by T the finite blow-up time of z.

In addition to the condition (3.2.38) we assume that the initial condition $\psi(x)$ satisfies

$$\psi = \psi(r) \in C^1(B)\,,\ \psi'(r) < 0 \text{ for } 0 < r < b \text{, and } \psi''(0) < 0\,. \tag{3.2.42}$$

In this case, the maximum principle applied to (3.2.34)-(3.2.36) establishes that $z(x,t)$ is also radial, i.e.

$$z = z(r,t)\,. \tag{3.2.43}$$

The problem (3.2.34)-(3.2.36) may then be written as

$$z_t = z_{rr} + \frac{(N-1)}{r} z_r + z^p - z^q |z_r| \qquad \text{in } B_T\,, \tag{3.2.44}$$

$$z(r,0) = \psi(r) \qquad \text{in } B\,, \tag{3.2.45}$$

$$z(r,t) = 0 \qquad \text{on } \partial B,\ 0 < t < T\,, \tag{3.2.46}$$

where $B_T = B \times (0,T)$, and we are able to verify the following Lemma.

Lemma 3.2.3

If $z(r,t)$ is a solution to (3.2.44)-(3.2.46) for which $\psi(r)$ satisfies (3.2.28) and (3.2.42) then

$$z_r < 0 \quad \text{in} \quad B_T \cap \{r > 0\}$$

for all $p > 1,\ q \geq 0$.

Proof

This result is established by arguments similar to those used in Lemma 2.1 of Friedman & MacLeod 1985 and begins by considering the function $w(r,t)$ defined as

$$w(r,t) = r^{N-1} z_r(r,t) \quad \text{in} \quad B_T \ . \tag{3.2.47}$$

Differentiating with respect to t we see that

$$\begin{aligned} w_t = w_{rr} - \frac{(N-1)}{r} w_r + p z^{p-1} w - \frac{q z^{q-1} w\,|\,w\,|}{r^{N-1}} \\ - z^q\, sign(z_r)\, w_r + \frac{(N-1)\, z^q\,|\,w\,|}{r} \end{aligned}$$

$$\text{in} \quad B_T \tag{3.2.48}$$

with $w(r,0) = r^{N-1}\psi'(r)$ in B (3.2.49)

and $w(r,t) = b^{N-1} z_r(b,t)$ on $\partial B,\ t \in (0,T)$. (3.2.50)

Hence, as $z = 0$ on $\partial B x (0,T)$ from (3.2.46), and $z>0$ within B_T , from the maximum principle applied to (3.2.44)-(3.2.46), it follows that

$$w(r,t) = b^{N-1} z_r(b,t) < 0 \quad \text{on} \quad \partial B \times (0,T)$$

and $w(r,0) = r^{N-1}\psi'(r) \le 0$ in B from (3.2.42)

and a positive maximum of w on the parabolic boundary of B_T is impossible.

Further, from (3.2.48) we see that

$$w_t - w_{rr} + \frac{(N-1)}{r} w_r - pz^{p-1}w + z^q \, sign(z_r) \, w_r$$
$$- \frac{(N-1) z^q |w|}{r} = - \frac{qz^{q-1}w |w|}{r^{N-1}}$$
$$\text{in } B_T \tag{3.2.51}$$

and if $p \ge 1, \ q \ge 0$, each of the coefficient on the left hand side of (3.2.51) must remain bounded throughout $B_T \cap \{r > 0\}$.

Hence, at a positive interior maximum of w we would have that

$$w_t - w_{rr} + \frac{(N-1)}{r} w_r - pz^{p-1}w + z^q \, sign(z_r) \, w_r$$
$$- \frac{(N-1) z^q | w |}{r} = - \frac{qz^{q-1}w | w |}{r^{N-1}} < 0,$$

a contradiction to the maximum principle.

We conclude therefore that

$$w = r^{N-1}z_r < 0 \qquad \text{in } B_T \cap \{r > 0\}$$

and hence Lemma 3.2.1.

We continue by assuming $z(r,t)$ is a solution to problem (3.2.44)-(3.2.46) for which Theorem 3.2.1 <u>does</u> apply and hence that $z(r,t)$ blows up at a finite time, say T , provided ψ is large enough compared to B and that condition (3.2.39) is satisfied.

As in Section 3.2.1 we define a function $\zeta(r,t)$, and regions R_1 and R_2 such that

$$\zeta(r,t) = z(0,t) \qquad \text{in } R_1 \tag{3.2.52}$$

$$\zeta(r,t) = z(r - \delta(t),t) \qquad \text{in } R_2 \tag{3.2.53}$$

where

$$R_1 = \{ 0 < r < \delta(t),\ 0 < t < T \} \tag{3.2.54}$$

$$R_2 = \{ \delta(t) < b + \delta(t),\ 0 < t < T \} \tag{3.2.55}$$

and in this case

$$\delta(t) = M(T - t) \tag{3.2.56}$$

for some positive constant M .

From Lemma 3.2.3 it is clear that

$$z(0,t) = \max_{x \in B} z(r,t) = m_z(t) \quad \text{say, for } 0 < t < T . \tag{3.2.57}$$

We propose that for a large number of functions $\varphi(x)$, $\zeta(r,t)$ will be a subsolution to $u(x,t)$, the solution to problem (3.2.1)-(3.2.4).

We verify this proposition as follows:-

In R_1 ,

$$\begin{aligned}\zeta_t - \nabla^2\zeta - \zeta^p + \zeta^\alpha |\nabla z|^\beta &= (z_t - \nabla^2 z - z^p + z^q |\nabla z|)\ (0,t) + z^\alpha |\nabla z|^\beta (0,t) \\ &= 0\end{aligned}$$

as $\nabla^2 z \le 0$ and $|\nabla z| = 0$ at $r = 0$.

Next, in R_2 ,

$$\begin{aligned}\zeta_t - \nabla^2\zeta - \zeta^p + \zeta^\alpha |\nabla\zeta|^\beta &= -\delta'(t)\, z_r + (N-1)\, z_r \left\{ \frac{1}{r-\delta} - \frac{1}{r} \right\} - z^q |z_r| + z^\alpha |z_r|^\beta \\ &\le |z_r| \left\{ z^\alpha |z_r|^{\beta-1} - z^q + \delta'(t) \right\}\end{aligned}$$

$$(3.2.58)$$

as $z_r < 0$ in R_2 from Lemma 3.2.3, and

$$\left\{ \frac{1}{r-\delta} - \frac{1}{r} \right\} > 0 \quad \text{for} \quad r > \delta .$$

Hence, as $\delta'(t) = -M$ from (3.2.56), it follows that

$$\zeta_t - \nabla^2\zeta - \zeta^p + \zeta^\alpha |\nabla\zeta|^\beta \le |z_r| \left\{ z^\alpha |z_r|^{\beta-1} - z^q - M \right\}$$

$$\text{in } R_2 . \qquad (3.2.59)$$

Condition (3.2.4) ensures that $\beta \ge 1$, and hence that, if the necessary conditions (i)-(iii) are satisfied, we may use inequality (3.2.41) to estimate

$$|z_r|^{\beta-1} \le \{ Cz^M + z^k - Az^l + B \}^{\beta-1}$$

for constants A, B and C, and $m = p-q > k > l > 1$.

Applying this estimate to the right hand side of (3.2.59) then yields that

$$\zeta_t - \nabla^2\zeta - \zeta^p + \zeta^\alpha |\nabla\zeta|^\beta \leq S \qquad \text{in} \quad R_2 \tag{3.2.60}$$

where

$$S = |z_r| \{ z^\alpha [Cz^m + z^k - Az^l + B]^{\beta-1} - z^q - M \}. \tag{3.2.61}$$

The right hand side of (3.2.61) will be less than or equal to zero in the following cases:-

Case A: if both z, and the constants A, B and C are small compared to M, or

Case B: if q is the highest power of z in the expression (3.2.61), which it will be if

$$\alpha + m(\beta-1) < q, \tag{3.2.62}$$

and if z is sufficiently large compared to A, B and C.

As we are free to choose the constant M, by choosing M large enough, it is possible to ensure that at least one of these cases will always be true and hence that S, and the right hand side of (3.2.60), is less than or equal to zero throughout R_2. This conclusion requires, however, that condition (3.2.62) and the necessary requirements of the estimate (3.2.9) (labelled (i)-(iii)) hold. As

from (iii), condition (3.2.62) reduces to

$m = p-q$

$$\alpha + (p-q)(\beta-1) < q \; . \tag{3.2.63}$$

We have already required in (3.2.39), that

$$0 \le q < p-1 \; ,$$

and therefore choose q 'close enough' to $p-1$. In this case, (3.2.63) <u>will</u> be satisfied if

$$\alpha+\beta < p \tag{3.2.64}$$

and the conditions (i)-(iii) required by the estimate (3.2.41) reduce to

(i) $\tfrac{1}{2}(p-1) > 0$

(ii) $p > \dfrac{2C}{2C-1}$

(iii) $C \ge 1$

and are automatic for $p > 1$ provided C is chosen suitably large.

To continue, therefore, as both ζ and $\nabla\zeta$ are continuous across the boundary $\partial R_1 \cap \partial R_2$, with $\nabla\zeta = \zeta_r = 0$ on this curve, we conclude that ζ satisfies the requirements necessary to be a subsolution to u in the interior, B' , of the set $\overline{R}_1 \cup \overline{R}_2$.

Further, ζ is continuous on the parabolic boundary of B', with

$$\zeta(r,0) = \psi(r) \qquad \text{for} \quad 0 \le r < b'$$

and $\quad \zeta = 0 \qquad$ elsewhere on $\partial_p B'$,

where $b' = b + \delta(0) = b + MT$.

It follows that if $B' \subset \Omega$, and if $\varphi(x) > \psi(x)$ for $x \in B'$, then by comparison

$$u(x,t) \ge \zeta(x,t) \tag{3.2.65}$$

for α, β and p satisfying (3.2.64) and (3.2.4).

In addition, Theorem 3.2.1 is valid for the considered function $z(r,t)$ so that z and hence ζ are functions which blow-up in a finite time.

We have therefore established the following Theorem.

Theorem 3.2.4

If $\varphi(x)$ is large enough compared to Ω (so that $\zeta(x,t)$ is a subsolution which blows up) and if α, β and p satisfy (3.2.4) and (3.2.64), then $u(x,t)$, the solution to (3.2.1)-(3.2.3) with $\varphi(x)$ satisfying (3.2.5) will blow-up in a finite time less than or equal to T .

This theorem represents a direct N-dimensional extension of Theorem 2.2.3 which, along with Remark 3.2.2 illustrates that the relationship between p and the sum $\alpha+\beta$ is critical in determining the behaviour of solutions to equations of this type. The preceeding analysis has established that, provided suitable initial data exists, the solution to (3.2.1)-(3.2.4) will remain finite for all time if

$$\alpha+\beta \geq p \ ,$$

and that solutions to this equation can exhibit finite-time blow-up if

$$\alpha+\beta < p \ .$$

It remains, therefore, to investigate the nature of this blow-up.

Section 3.3 Identification of the blow-up sets

In this section, we consider a solution, $u(x,t)$, to the problem (3.2.1)-(3.2.4) which satisfies the requirements of Theorem 3.2.4 and which consequently blows up at a finite time, say T.

Once blow-up has been established, one of the questions raised is of where the blow-up takes place, and in particular, when is the blow-up confined to a single point.

The form of equation (3.2.1) makes this question even more intriguing. Although the gradient term has a damping effect and works against blow-up (to the extent that it may even ensure that blow-up does not take place, e.g. Remark 3.2.2), if blow-up does occur, then the singularising effect of the gradient term may be expected to play an important role in determining where blow-up takes place. It is consequently not obvious if a solution to (3.2.1)-(3.2.4) exhibits anything like the blow-up behaviour it would if the gradient term were absent, i.e. if it were a solution to the type of equation considered in Friedman & MacLeod 1985.

The purpose of this section is to investigate the blow-up sets associated with solutions to the problem (3.2.1)-(3.2.4) to which Theorem 3.2.4 applies. This analysis can be simplified by initially considering the symmetric problem to which the techniques developed in the study of the one-dimensional problem of Chapter 2 may be extended. This approach also serves to indicate what results may be available in the more general higher dimensional case, and the symmetric problem is consequently addressed in the following subsection.

3.3.1 The symmetric problem

In this subsection we consider the problem (3.2.1)-(3.2.4) in the case that

Ω is the ball, $B_R = \{ x: |x| < R \},$ (3.3.1)

and where the initial condition, $\varphi(x)$, satisfies

$\varphi = \varphi(r)$ with $\varphi'(r) < 0$ for $0 < r = |x| \le R$ and $\varphi''(0) < 0,$ (3.3.2)

in addition to (3.2.5).

It follows, therefore, that u is also radial, i.e.

$$u = u(r, t). \qquad (3.3.3)$$

In this case, (3.2.1)-(3.2.4) can be written as

$$u_t = u_{rr} + \frac{(N-1)}{r} u_r + u^p - u^\alpha |u_r|^\beta \quad \text{in} \quad B_T, \qquad (3.3.4)$$

$$u(r,0) = \varphi(r) \quad \text{in} \quad B_R, \qquad (3.3.5)$$

$$u(R,t) = 0 \quad \text{for} \quad t > 0, \qquad (3.3.6)$$

where

$$p > 1, \ \alpha \ge 0 \ \text{and} \ \beta \ge 1, \qquad (3.3.7)$$

and where B_T denotes $B_R x (0,T)$.

We begin by extending Lemma 3.2.3 to the solution, $u(r,t)$, of problem (3.3.4)-(3.3.7).

Lemma 3.3.1

If the initial condition $\varphi = \varphi(r)$ satisfies conditions (3.2.5) and (3.3.2), then

$u(r,t)$, the solution to (3.3.4)-(3.3.7), satisfies

$$u_r < 0 \quad \text{in} \quad B_T \cap \{r > 0\}.$$

Proof

The proof of this result is similar to that of Lemma 3.2.3 and is consequently not repeated.

If we define a function $w(r,t)$ as

$$w(r,t) = r^{N-1} u_r(r,t) \quad \text{in} \quad B_T,$$

then we see that

$$w < 0 \quad \text{in} \quad B_T \cap \{r > 0\}$$

in light of Lemma 3.3.1.

We next consider the function $J(r,t)$, where

$$J(r,t) = w(r,t) + \epsilon c(r) g(u(r,t)) \quad \text{in} \quad B_T, \tag{3.3.8}$$

where ϵ is a small positive number and $c(r), g(u)$ are positive functions for which

$$c(0) = g(0) = 0. \tag{3.3.9a}$$

It is proposed that $J \le 0$ in B_T if ϵ is small enough and $c(r)$ and $g(u)$ are chosen appropriately. To establish this proposition, we see from (3.3.8) that

$$\begin{aligned} J(R,t) &= w(R,t) + \epsilon c(R) g(0) \\ &= w(R,t) \\ &\le 0 \end{aligned} \tag{3.3.9}$$

from (3.3.6), (3.3.9a) and Lemma 3.3.1

$$\begin{aligned} J(0,t) &= w(0,t) + \epsilon c(0) g(u) \\ &= w(0,t) \\ &= 0 \end{aligned} \tag{3.3.10}$$

by (3.3.9a) and Lemma 3.3.1, and

$$\begin{aligned} J(r,0) &= r^{N-1}\varphi'(r) + \epsilon c(r) g(\varphi) \\ &\le 0 \qquad \text{for } 0 \le r \le R \end{aligned} \tag{3.3.11}$$

in light of (3.3.9a) and (3.3.2) provided ϵ is chosen suitably small.

Next, on differentiating (3.3.8)

$$J_t = w_t + \epsilon c g' u_t \tag{3.3.12}$$

$$J_r = w_r + \epsilon c' g + \epsilon c g' u_r \tag{3.3.13}$$

$$J_{rr} = w_{rr} + \epsilon c'' g + 2\epsilon c' g' u_r + \epsilon c g'' u_r^2 + \epsilon c g' u_{rr}, \tag{3.3.14}$$

so that, as $w \le 0$ in B_T , and $u_r = w/r^{N-1},$ we find that

$$
\begin{aligned}
J_t - J_{rr} = &- \frac{(N-1)\,w_r}{r} + pu^{p-1}w - \frac{\alpha u^{\alpha-1}|w|^{\beta}w}{r^{\beta(N-1)}} \\
&+ \frac{\beta u^{\alpha}|w|^{\beta-1}w_r}{r^{(\beta-1)(N-1)}} + \frac{(N-1)\,\beta u^{\alpha}|w|^{\beta}}{r^{(\beta-1)(N-1)+1}} + \frac{\epsilon c g'(N-1)\,w}{r^{N-2}} \\
&+ \epsilon c g' u^{p} - \frac{\epsilon c g' u^{\alpha}|w|^{\beta}}{r^{\beta(N-1)}} - \epsilon c'' g \\
&- \frac{2\epsilon c' g' w}{r^{N-1}} - \frac{\epsilon c g'' w^2}{r^{2(N-1)}}.
\end{aligned}
$$

(3.3.15)

Using (3.3.13) to substitute for w_r and (3.3.8) to substitute for w in (3.3.15)

and if

$$g''(u) \geq 0 \quad \text{in} \quad B_T \tag{3.3.16}$$

we find that

$$J_t - J_{rr} + aJ_r + bJ \leq S \tag{3.3.17}$$

where

$$
\begin{aligned}
S = &\frac{\epsilon(N-1)\,c'g}{r} + \frac{2\epsilon^2 c g g'}{r^{(N-1)}}\left\{ c' - \frac{c(N-1)}{r} \right\} - \epsilon c\,\{ pu^{p-1}g - g'u^{p} \} \\
&+ \frac{\epsilon^{\beta+1}c^{\beta+1}g^{\beta}u^{\alpha-1}}{r^{\beta(N-1)}}\,\{ \alpha g + (\beta-1)\,g'u \} - \epsilon c'' g \\
&- \frac{\epsilon^{\beta}c^{\beta-1}\beta u^{\alpha}g^{\beta}}{r^{(\beta-1)(N-1)}}\left\{ c' - \frac{c(N-1)}{r} \right\}.
\end{aligned}
$$

(3.3.18)

In addition, we have, from (3.3.10), that $J = 0$ at $r = 0$ and a positive maximum of J is therefore impossible at this point. Further, from (3.3.8) and (3.3.9),

$$
\begin{aligned}
J(r,t) &= w(r,t) + \epsilon c(r)\,g(u(r,t)) \\
&= w(r,t) < 0 \quad \text{throughout} \quad B_T
\end{aligned}
$$

at any point at which $u(r,t) = 0$.

Hence, as u cannot be zero, and as J is clearly non-zero at any point within B_T at which J is positive, it follows that the functions a and b appearing in (3.3.17) must remain bounded at any positive interior maximum of J.

If we can demonstrate, therefore, that S (as described by (3.3.18)) is less than or equal to zero throughout B_T then the maximum principle applied to (3.3.17) would allow the conclusion that a positive maximum of J is impossible at any interior point of B_T.

To establish this requirement we choose

$$c(r) = r^{\gamma} \quad \text{and} \quad g(u) = u^{k} \tag{3.3.19}$$

for some positive γ and $k > 1$. This choice ensures that the conditions (3.3.9a) and (3.3.16) are satisfied as required. Equation (3.3.18) now becomes

$$\begin{aligned} S = \epsilon r^{\gamma} \{ \, & \gamma(N-\gamma)u^{k}/r^{2} + 2\epsilon(\gamma-N+1)kr^{\gamma-N}u^{2k-1} \\ & - (p-k)u^{p+k-1} + \epsilon^{\beta}(\alpha+k(\beta-1))r^{\beta(\gamma-N+1)}u^{\alpha-1+k(\beta+1)} \\ & - \epsilon^{\beta-1}\beta(\gamma-N+1)r^{(\beta-1)(\gamma-N+1)-1}u^{\alpha+k\beta} \, \}, \end{aligned} \tag{3.3.20}$$

from which we see that S will be less than or equal to zero if

$$\begin{aligned} & (p-k)u^{p+k-1} + \gamma(\gamma-N)u^{k}/r^{2} + \epsilon^{\beta-1}\beta(\gamma-N+1)r^{(\beta-1)(\gamma-N+1)-1}u^{\alpha+k\beta} \\ & \quad \geq 2\epsilon(\gamma-N+1)kr^{\gamma-N}u^{2k-1} + \epsilon^{\beta}(\alpha+k(\beta-1))r^{\beta(\gamma-N+1)}u^{\alpha-1+k(\beta+1)}. \end{aligned} \tag{3.3.21}$$

If we assume that $p > k$ and $\gamma > N$, then inequality (3.3.21) will be satisfied if either

$$(p-k)\, u^{p+k-1} \geq (\alpha+k(\beta-1))\,\epsilon^{\beta} r^{\beta(\gamma-N+1)} u^{\alpha-1+k(\beta+1)} + 2\epsilon(\gamma-N+1)\, k r^{\gamma-N} u^{2k-1}$$

which we call condition A, or if

$$\gamma(\gamma-N)\, u^{k}/r^{2} \geq (\alpha+k(\beta-1))\,\epsilon^{\beta} r^{\beta(\gamma-N+1)} u^{\alpha-1+k(\beta+1)} + 2\epsilon(\gamma-N+1)\, k r^{\gamma-N} u^{2k-1},$$

which we call condition B.

Inequality (3.3.21) can now be established by applying arguments similar to those used in the study of the one-dimensional problem.

We begin by observing that condition A will be satisfied if

$$p + k - 1 > \alpha - 1 + k(\beta + 1) \quad \text{and} \quad p + k - 1 > 2k - 1$$

with $\quad \tfrac{1}{2}(p-k)\, u^{p+k-1-(\alpha-1+k(\beta+1))} > (\alpha + k(\beta - 1))\,\epsilon^{\beta} r^{\beta(\gamma-N+1)},$

and

$$\tfrac{1}{2}(p - k)\, u^{p+k-1-(2k-1)} > 2\epsilon(\gamma - N + 1)\, k r^{\gamma-N}.$$

As we have assumed that $\gamma > N$, this requires that

$$(p - \alpha)/\beta > k \quad \text{and} \quad p > k,$$

with u 'large enough' compared to ϵ.

Alternatively, condition B will be satisfied if

$$k < \alpha - 1 + k(\beta+1) \quad \text{and} \quad k < 2k - 1$$

with $\tfrac{1}{2}\gamma(\gamma - N) > (\alpha + k(\beta - 1)\epsilon^{\beta} r^{(\gamma+3-N)} u^{\alpha-1+k(\beta+1)-k}$

and $\tfrac{1}{2}\gamma(\gamma - N) > 2\epsilon(\gamma + 1 - N) k r^{\gamma+2-N} u^{2k-1-k}$

i.e. for $\gamma > N,$ if

$$\alpha + k\beta > 1 \quad \text{and} \quad k>1$$

with u 'small enough' compared to $1/\epsilon$.

It follows, therefore, that if k can be chosen such that

$$(p-\alpha)/\beta > k > 1,\ p > k \quad \text{and} \quad \alpha + k\beta > 1 \tag{3.3.22}$$

with γ 'large' (specifically $\gamma > N$) and ϵ sufficiently small, then at least one of conditions A or B will always be satisfied, and consequently that $S,$ as defined by (3.3.20), will be no greater than zero as required.

Hence, from the maximum principle applied to (3.3.17) we see that a positive maximum of J is only possible on the parabolic boundary of $B_T,$ although (3.3.9)-(3.3.11) ensures that J can never be greater than zero at such points.

We conclude, therefore, that

$$J(r,t) - r^{N-1}u_r(r,t) + \epsilon r^{\gamma}u^{k}(r,t) \leq 0 \quad \text{in} \quad B_T, \tag{3.3.23}$$

provided γ is 'large', ϵ is 'small', and condition (3.3.22) is satisfied.

We are now able to describe the following Theorem.

Theorem 3.3.2

If $u(r,t)$ is a solution to (3.3.5)-(3.3.7) to which Theorem 3.2.4 applies and if, in addition to the conditions required by Theorem 3.2.4, $\varphi(x)$ satisfies (3.3.2), then u blows up at the single point $x = 0$.

Proof

Theorem 3.2.4 ensures that u blows up at the finite time denoted by T.

From inequality (3.3.23) it follows that

$$-r^{N-1}u_r(r,t) \geq \epsilon r^{\gamma} u^{k}(r,t) \quad \text{in } B_T \tag{3.3.24}$$

for small ϵ, $\gamma > N$ and some $k > 1$, provided $k > 1$ can be chosen to satisfy (3.3.22).

If inequality (3.3.24) is integrated from 0 to r for any $0 < r < R$, then as $k > 1$ and $u(0,t) \geq c > 0$, it follows that there exists a constant $C > 0$ for which

$$\frac{1}{(k-1)\,u^{k-1}(r,t)} \geq \epsilon C r^{\gamma+2-N}. \tag{3.3.25}$$

Hence, if there exists some $r > 0$ for which $u(r,t) \to \infty$ as $t \to T$, i.e. if there exists a blow-up point (r,T) for which $r \neq 0$, this would lead to a contradiction to (3.3.25).

Finally, we note that the inequality (3.3.23) requires that some $k > 1$ can be chosen to satisfy the conditions (3.3.22). However, as we have assumed that Theorem 3.2.4 applies it follows that

$$p > \alpha + \beta \quad \text{and} \quad \alpha + \beta \geq 1$$

(from (3.2.4) and (3.2.64)) and hence that the existence of a suitable $k > 1$ to satisfy (3.3.22) is automatic.

As indicated in the introduction to this section, we have only been able to establish single-point blow-up in the N-dimensional case for the symmetric problems described. However, in the following subsection, we consider a more general N-dimensional form of the problem (3.2.1)-(3.2.4) for which useful information as to the size and location of the blow-up sets may still be available.

3.3.2 The non-symmetric problem

In this section, we extend our study to a more general from of the problem (3.2.1)-(3.2.4) although we restrict ourselves to the situation in which

$$\Omega \text{ is a convex domain in } \mathbf{R}^N. \tag{3.3.26}$$

Throughout this section, we shall assume that the considered solution to problem (3.2.1)-(3.2.4) satisfies the requirements Theorem 3.2.4.

Hence, this function will blow-up at a finite time which we denote by T.

For this more general problem, the determination of any information regarding either the size or the location of the blow-up set is less straightforward. In particular, neither the one-dimensional arguments of Section 2.3, nor the techniques of the preceding sections (which are essentially identical) may be applied to identify the blow-up set so precisely. We proceed, therefore, to estimate as far as possible, where blow-up may occur by applying the following extension of Theorem 3.3 from Friedman & MacLeod 1985.

If ν is any outward pointing normal to $\partial\Omega$ we assume that

$$\frac{\partial\varphi}{\partial\nu} < 0 \quad \text{on} \quad \partial\Omega \ . \tag{3.3.27}$$

We next consider any $y_0 \in \partial\Omega$ and for simplicity set $y_0 = 0$ and take the half-space $\{ x_1 > 0 \}$ to be tangent to Ω at y_0 with $x_1 < 0$ for any $x = (x_1, x_2, \ldots x_N) \in \Omega$.

For some $\delta < 0$ with $|\delta|$ small we define the sets Ω_δ^+ and Ω_δ^- as

$$\Omega_\delta^+ = \Omega \cap \{ x_1 > \delta \} \tag{3.3.28}$$

and $$\Omega_\delta^- = \{ (x_1, x') : (2\delta - x_1, x') \in \Omega_\delta^+ \} \tag{3.3.29}$$

where $x' = (x_2, x_3 \ldots x_N)$. Hence Ω_δ^- is the reflection of Ω_δ^+ with respect to the plane $\{ x_1 = \delta \}$.

We next consider a function $w(x,t)$ defined as

$$w(x,t) = u(x_1, x', t) - v(x_1, x', t) \quad \text{for } x \text{ in } \Omega_\delta^-, 0 < t < T \tag{3.3.30}$$

where

$$v(x_1, x', t) = u(2\delta - x_1, x', t)\,. \tag{3.3.31}$$

On differentiating (3.3.30) we find that $w(x,t)$ satisfies

$$w_t = \nabla^2 w + cw - du^\alpha |\nabla w| - (u^\alpha - v^\alpha)\, |\nabla v|^\beta \tag{3.3.32}$$

where

$$c = \frac{(u^p - v^p)}{(u - v)}, \quad \text{and} \quad d = \frac{|\nabla u|^\beta - |\nabla v|^\beta}{|\nabla u| - |\nabla v|},$$

are, in light of (3.2.4), bounded near points at which $u = v$ and/or $|\nabla u| = |\nabla v|$.

Hence, as

$$-(u^\alpha - v^\alpha)\, |\nabla v|^\beta > 0$$

at any negative minimum of w, it follows that

$$w_t - \nabla^2 w - cw + du^\alpha |\nabla w| > 0 \tag{3.3.33}$$

at such a point which is a contradiction to the maximum principle.

Further,

$$w = 0 \quad \text{on} \quad \{ x_1 = \delta \} \tag{3.3.34}$$

and

$$w = u(x_1, x', t) > 0 \quad \text{on} \quad (\partial\Omega_\delta^- \cap \{x_1 < \delta\}) x (0, T)\,. \tag{3.3.35}$$

Finally, in light of (3.3.27)

$$w(x,0) > 0 \quad \text{for} \quad x \in \Omega_\delta^- \tag{3.3.35}$$

provided $|\delta|$ is sufficiently small.

We conclude, therefore, from (3.3.33)-(3.3.35) that

$$w > 0 \quad \text{in} \quad \Omega_\delta^- x (0, T), \tag{3.3.37}$$

and

$$0 > \frac{\partial w}{\partial x_1} = 2 \frac{\partial u}{\partial x_1}, \quad \text{on} \quad \{ x_1 = \delta \}. \tag{3.3.38}$$

Also, as δ is arbitrary, it follows by varying δ that

$$\frac{\partial u}{\partial x_1} < 0 \quad \text{for} \quad x \in \Omega_{\delta_0}^+, 0 < t < T \tag{3.3.39}$$

provided $|\delta_0|$ is sufficiently small.

We now consider the function $J(x,t)$ defined as

$$J(x,t) = u_{x_1}(x,t) + \epsilon c(x_1) g(u) \quad \text{in} \quad \Omega_{\delta_0}^+ x (0, T) \tag{3.3.40}$$

for ϵ some small positive constant, and $c(x_1)$ and $g(u)$ some positive

functions yet to be determined.

On differentiating (3.3.40) we see that

$$J_t = u_{x_1 t} + \epsilon c g' u_t \tag{3.3.41}$$

$$\nabla J = \nabla u_{x_1} + \epsilon \nabla c g + \epsilon c g' \nabla u \tag{3.3.42}$$

$$\nabla^2 J = \nabla^2 u_{x_1} + \epsilon \nabla^2 c g + 2\epsilon g' \nabla c \cdot \nabla u + \epsilon c g'' |\nabla u|^2 + \epsilon c g' \nabla^2 u, \tag{3.3.43}$$

so that

$$\begin{aligned} J_t - \nabla^2 J = p u^{p-1} u_{x_1} - \alpha u^{\alpha-1} |\nabla u|^{\beta} u_{x_1} - \beta u^{\alpha} |\nabla u|^{\beta-2} \nabla u \cdot \nabla u_{x_1} \\ + \epsilon c g' u^p - \epsilon c g' u^{\alpha} |\nabla u|^{\beta} - \epsilon \nabla^2 c g \\ - 2\epsilon g' \nabla u \cdot \nabla c - \epsilon c g'' |\nabla u|^2 . \end{aligned} \tag{3.3.44}$$

If (3.3.42) is used to substitute for ∇u_{x_1}, in the right hand side of (3.3.44) we find that

$$\begin{aligned} J_t - \nabla^2 J + \beta u^{\alpha} |\nabla u|^{\beta-2} \nabla u \cdot \nabla J \\ = p u^{p-1} u_{x_1} - \alpha u^{\alpha-1} |\nabla u|^{\beta} u_{x_1} + \epsilon \beta u^{\alpha} |\nabla u|^{\beta-2} \nabla u \cdot \nabla c g \\ + \epsilon (\beta - 1) u^{\alpha} |\nabla u|^{\beta} c g' + \epsilon c g' u^p - \epsilon \nabla^2 c g \\ - 2\epsilon g' \nabla u \cdot \nabla c - \epsilon c g'' |\nabla u|^2 . \end{aligned} \tag{3.3.45}$$

As $c = c(x_1)$, equation (3.3.45) reduces to

$$\begin{aligned} J_t - \nabla^2 J + \beta u^{\alpha} |\nabla u|^{\beta-2} \nabla u \cdot \nabla J \\ = p u^{p-1} u_{x_1} - \alpha u^{\alpha-1} |\nabla u|^{\beta} u_{x_1} + \epsilon \beta u^{\alpha} |\nabla u|^{\beta-2} c' g u_{x_1} \\ + \epsilon c (\beta - 1) u^{\alpha} |\nabla u|^{\beta} g' + \epsilon c g' u^p - \epsilon c'' g \\ - 2\epsilon c' g' u_{x_1} - \epsilon c g'' |\nabla u|^2 . \end{aligned} \tag{3.3.46}$$

If (3.3.40) is now used to substitute for u_{x_1} in the right hand side of equation (3.3.46) we find that

$$J_t - \nabla^2 J + b \cdot \nabla J + d J = S \tag{3.3.47}$$

where

$$
\begin{aligned}
S = &- \epsilon c\,\{\, p u^{p-1} g - u^{p} g' \,\} + \epsilon c u^{\alpha-1} \,|\, \nabla u \,|^{\beta} \{\, \alpha g + (\beta-1) u g' \,\} \\
&- \epsilon^{2} c c' \beta u^{\alpha} \,|\, \nabla u \,|^{\beta-2} g^{2} - \epsilon c'' g + 2\epsilon^{2} c c' g g' \\
&- \epsilon c g'' \,|\, \nabla u \,|^{2} .
\end{aligned}
\tag{3.3.48}
$$

From (3.3.39) we see that $u_{x_1} < 0$ throughout $\Omega^{+}_{\delta_0} \times (0,T)$ and hence that $|\nabla u| = 0$ is impossible within the interior of this set.

Further, if $g(u)$ is chosen such that

$$
g(0) = 0, \tag{3.3.49}
$$

then if $(\overline{x},\overline{t})$ is any point at which $u(\overline{x},\overline{t}) = 0,$ it follows that

$$
\begin{aligned}
J(\overline{x},\overline{t}) &= u_{x_1}(\overline{x},\overline{t}) + \epsilon c(\overline{x}_1)\, g(u(\overline{x},\overline{t})) \\
&= u_{x_1}(\overline{x},\overline{t}) + \epsilon c(\overline{x}_1)\, g(0) \\
&= u_{x_1}(\overline{x},\overline{t}) < 0 \text{ by (3.3.39)}.
\end{aligned}
\tag{3.3.50}
$$

Hence, it is also impossible for J to have a positive maximum within $\Omega^{+}_{\delta_0} \times (0,T)$ at any point at which $u = 0$. Finally, as $J \neq 0$ at a positive maximum of $J,$ it is clear, in light of (3.2.4), that the functions b and d appearing in the left hand side of equation (3.3.47) are bounded at any positive maximum of $J(x,t)$ within $\Omega^{+}_{\delta_0} \times (0,T)$.

We next choose the functions $c(x_1)$ and $g(u)$ as

$$c(x_1) = (x_1-\delta_0)^{\gamma} \quad \text{and} \quad g(u) = u^{s}$$

for some positive constants γ and s, and this ensures g satisfies condition (3.3.49) as required. With this choice of c and g, the equation (3.3.48) for S becomes

$$\begin{aligned} S = -\epsilon c\{ &(p-s)u^{p+s-1} - (\alpha+s(\beta-1))u^{\alpha+s-1}|\nabla u|^{\beta} \\ &+ \epsilon\gamma(x_1-\delta_0)^{\gamma-1}\beta u^{\alpha}|\nabla u|^{\beta-2}u^{2s} \\ &+ \frac{\gamma(\gamma-1)u^{s}}{(x_1-\delta_0)^2} - 2\epsilon\gamma(x_1-\delta_0)^{\gamma-1}su^{2s-1} \\ &+ s(s-1)u^{s-2}|\nabla u|^2\}. \end{aligned} \tag{3.3.51}$$

In light of equation (3.3.47) we see that, if it can be established that S as given by equation (3.3.51) is less than or equal to zero throughout $\Omega^{+}_{\delta_0}\times(0,T)$, then the maximum principle would yield the conclusion that a positive maximum of J is impossible within the interior of this set.

The verification of this requirement for all α, β and p considered (i.e. all α, β and p for which Theorem 3.2.4 applies) is investigated. This analysis is similar to that of Section 2.4 in which the sign of the term described by equation (2.4.20) is considered.

The presence of the gradient terms in (3.3.51), however, means that the sign of equation (3.3.51) cannot be directly related to the term in (2.4.20) and we must again begin at first principles although reference will be made to the arguments of

Section 2.4 where convenient. We consider three cases in each of which we seek to demonstrate that s, as defined by equation (3.3.51), is less than or equal to zero.

This is clearly true if the following inequality is satisfied

$$(p-s)u^{p+s-1} + \frac{\gamma(\gamma-1)u^s}{(x-\delta_0)^2} + s(s-1)u^{s-2}|\nabla u|^2$$
$$\geq (\alpha+s(\beta-1))u^{\alpha+s-1}|\nabla u|^\beta + 2\epsilon\gamma(x_1-\delta_0)^{\gamma-1}su^{2s-1}. \tag{3.3.52}$$

Case 1 $\beta < \frac{2(p-\alpha)}{(p+1)}$

If $$s(s-1)u^{s-2}|\nabla u|^2 \geq (\alpha+s(\beta-1))u^{\alpha+s-1}|\nabla u|^\beta \tag{3.3.53}$$

then inequality (3.3.52) will be satisfied if

$$(p-s)u^{p+s-1} \geq 2\epsilon\gamma(x_1-\delta_0)^{\gamma-1}su^{2s-1}$$

i.e. if $p > s$ and $\gamma \geq 1$ and provided u is 'large enough' compared to $\epsilon\gamma$. We call this condition A.

If inequality (3.3.53) holds, however, (3.3.52) will also be satisfied if

$$\frac{\gamma(\gamma-1)u^s}{(x_1-\delta_0)^2} \geq 2\epsilon\gamma(x_1-\delta_0)^{\gamma-1}su^{2s-1}$$

i.e. if

$$s > 1, \gamma \geq 1$$

and provided u is 'small enough' compared to γ/ϵ.

We call this condition B.

It follows that if $\gamma \geq 1$ and

$$p > s > 1 \text{ with } \epsilon \text{ suitably small} \tag{3.3.54}$$

then at least one of conditions A or B will always be satisfied, and hence that inequality (3.3.52) holds if inequality (3.3.53) is true.

Alternatively, if inequality (3.3.53) is not true, i.e. if

$$|\nabla u|^{2-\beta} \leq \left\{\frac{(\alpha+s(\beta-1))}{s(s-1)}\right\} u^{\alpha+1} \tag{3.3.55}$$

then as $\beta < 2$ (because $\beta < 2(p-\alpha)/(p+1) < 2$) we may use (3.3.55) to estimate for the $|\nabla u|^{\beta}$ term in the right hand side of (3.3.52). Hence, if (3.3.53) is false, (3.3.52) will be satisfied if

$$(p-s)\,u^{p+s-1} + \frac{\gamma(\gamma-1)\,u^{s}}{(x_1-\delta_0)^2} \geq D u^{\alpha+s-1+\beta(\alpha+1)/(2-\beta)} + 2\epsilon\gamma\,(x_1-\delta_0)^{\gamma-1} s u^{2s-1}, \tag{3.3.56}$$

where

$$D = \frac{(\alpha+s(\beta-1))^{1+\beta/(2-\beta)}}{(s(s-1))^{\beta/(2-\beta)}},$$

and we have dropped the (positive) gradient term from the left hand side of (3.3.52), under the assumption that $s > 1$.

Inequality (3.3.56) will also be satisfied in two cases; if

$$(p-s)\,u^{p+s-1} \geq Du^{\alpha+s-1+\beta(\alpha+1)/(2-\beta)} + 2\epsilon\gamma\,(x_1-\delta_0)^{\gamma-1}su^{2s-1},$$

which we call condition C, or if

$$\frac{\gamma(\gamma-1)\,u^{s}}{(x_1-\delta_0)^{2}} \geq Du^{\alpha+s-1+\beta(\alpha+1)/(2-\beta)} + 2\epsilon\gamma\,(x_1-\delta_0)^{\gamma-1}su^{2s-1}$$

which we call condition D.

Condition C will be satisfied if

$$p + s - 1 > \alpha + s - 1 + \beta(\alpha+1)/(2-\beta) \quad \text{and} \quad p + s - 1 > 2s - 1$$

and if u is 'large enough' compared to D and $\epsilon\gamma$.

We note, however, that the first of these inequalities, i.e.

$$p + s - 1 > \alpha + s - 1 + \beta(\alpha+1)/(2-\beta)$$

is automatic as we have assumed $\beta < 2(p-\alpha)/(p+1)$.

Alternatively, condition D will be satisfied if

$$s \leq \alpha + s - 1 + \beta(\alpha+1)/(2-\beta) \quad \text{and} \quad s \leq 2s-1, \tag{3.3.58}$$

and if u is 'small enough' compared to γ and γ/ϵ, or, if there is equality in either part of (3.3.58), simply if γ and γ/ϵ are sufficiently large.

As $s > 1$, the second inequality in (3.3.58) is automatic and the first inequality in (3.3.58) will be satisfied if

$$\alpha + \beta(\alpha+1)/(2-\beta) \geq 1, \quad \text{i.e. if} \quad \alpha + \beta \geq 1$$

which is also true in the light of (3.2.4).

We conclude, therefore, that if ϵ and $1/\gamma$ are sufficiently small, then at least one of conditions C or D will always be true, and hence that inequality (3.3.52) is satisfied when (3.3.53) is false provided

$$p + s - 1 > 2s - 1 \quad \text{and} \quad s > 1,$$

i.e. provided

$$p > s > 1.$$

On combining these results we see that, if $\beta < 2(p-\alpha)/(p+1)$ then inequality (3.3.52) is satisfied provided $p > s > 1$ with $\gamma \geq 1$ and ϵ and $1/\gamma$ suitably small.

Case 2 $\beta = 2(p-\alpha)/(p+1)$

If $\beta = 2(p-\alpha)/(p+1)$, then the arguments of Case 1 may be repeated identically to the conclusion that inequality (3.3.52) will be satisfied if inequality (3.3.53) holds and, for $\gamma \geq 1$, if (3.3.54) is satisfied, or alternately, if (3.3.53) is false and either of the conditions C or D hold.

Condition C requires that

$$(p-s)\,u^{p+s-1} \geq Du^{\alpha+s-1+\beta(\alpha+1)/(2-\beta)} + 2\epsilon\gamma\,(x_1-\delta_0)^{\gamma-1} s u^{2s-1} \ ,$$

and condition D that

$$\frac{\gamma(\gamma-1)\,u^{s}}{(x_1-\delta_0)^2} \geq Du^{\alpha+s-1+\beta(\alpha+1)/(2-\beta)} + 2\epsilon\gamma\,(x_1-\delta_0)^{\gamma-1} s u^{2s-1} \ ,$$

where

$$D = \frac{(\alpha+s(\beta-1))^{1+\beta/(2-\beta)}}{(s(s-1))^{\beta/(2-\beta)}} \ .$$

For $\beta = 2(p-\alpha)/(p+1)$ we see that

$$p + s - 1 = \alpha + s - 1 + \beta(\alpha+1)/(2-\beta)$$

and hence that condition C will only be satisfied

if

$$(p-s) > D\,, \tag{3.3.59}$$

and if

$$\{(p-s) - D\}\, u^{p+s-1} \geq 2\epsilon\gamma\,(x_1-\delta_0)^{\gamma-1} s u^{2s-1}. \tag{3.3.60}$$

Assuming for the moment that condition (3.3.59) is true, we may proceed as before to conclude that condition D will be satisfied if

$$s > 1$$

and if u is small enough compared to γ and γ/ϵ , or simply if γ and γ/ϵ are sufficiently large in the case $\alpha + \beta - 1$.

We may also proceed, under the assumption that condition (3.3.59) is true, to conclude that inequality (3.3.60) and hence condition C will be satisfied if

$$p > s$$

and if u is 'large enough' compared to $\epsilon\gamma$.

It follows, therefore, that if inequality (3.3.59) can be established, then by assuming ϵ and $1/\gamma$ are sufficiently small, inequality (3.3.52) may again be verified, for $\beta - 2(p-\alpha)/(p+1)$, provided

$$p > s > 1\ .$$

We therefore proceed to investigate the inequality (3.3.59). On substituting for D by (3.3.57) inequality (3.3.59) becomes

$$(p-s) > \frac{(\alpha+s(\beta-1))^{1+\beta/(2-\beta)}}{(s(s-1))^{\beta/(2-\beta)}} . \tag{3.3.61}$$

With $\beta = 2(p-\alpha)/(p+1)$, it follows that

$$\frac{\beta}{(2-\beta)} = \frac{(p-\alpha)}{(\alpha+1)} , \quad 1 + \frac{\beta}{(2-\beta)} = \frac{(p+1)}{(\alpha+1)} , \text{ and } \quad \beta - 1 = \frac{p-(2\alpha+1)}{(p+1)}$$

and inequality (3.3.61) may be written as

$$(p-s)\{s(s-1)\}^{(p-\alpha)/(\alpha+1)} > \left\{\alpha + \frac{s(p-(2\alpha+1))}{(p+1)}\right\}^{(p+1)/(\alpha+1)} . \tag{3.3.62}$$

We next choose

$$s = (p-\alpha)/\beta = \tfrac{1}{2}(p+1) \tag{3.3.63}$$

so that

$$p > s > 1$$

is automatic as $p > 1$.

With this s , (3.3.62) becomes

$$\tfrac{1}{2}(p-1)\{\tfrac{1}{2}(p+1)\tfrac{1}{2}(p-1)\}^{(p-\alpha)/(\alpha+1)} > \{\tfrac{1}{2}(p-1)\}^{(p+1)/(\alpha+1)} .$$

As $p > 1$ and $p > \alpha + \beta > \alpha$ (from the requirements of Theorem 3.2.4), this is given by

$$\tfrac{1}{2}(p+1) > 1 \text{ , i.e. } p > 1$$

which is also true.

We conclude that, if $\beta = 2(p-\alpha)/(p+1)$, then inequality (3.3.52) is again satisfied if ϵ and $1/\gamma$ are suitably small, and if

$$s = (p-\alpha)/\beta = \tfrac{1}{2}(p+1) \ .$$

Case 3 $\beta > 2(p-\alpha)/(p+1)$

It remains, finally, to consider the case of $\beta > 2(p-\alpha)/(p+1)$. To recap, inequality (3.3.52) requires that

$$(p-s)\,u^{p+s-1} + \frac{\gamma(\gamma-1)\,u^{s}}{(x_1-\delta_0)^2} + s(s-1)\,u^{s-2}\,|\nabla u|^2$$
$$\geq (\alpha+s(\beta-1))u^{\alpha+s-1}\,|\nabla u|^{\beta} + 2\epsilon\gamma\,(x_1-\delta_0)^{\gamma-1}su^{2s-1}.$$

In this case, as $\beta > 2(p-\alpha)/(p+1)$, and, as we have taken Ω to be a convex domain (from (3.3.26)) we may make use of the upper bound for $|\nabla u|$ developed in Appendix A. Hence we see that

$$|\nabla u|^2 \le \{ Cu^m + u^k - Au^l + B \}^2 \tag{3.3.64}$$

for appropriate constant A, B and C, provided

(i) $\beta > 2(p-\alpha)/(p+1)$,

(ii) $C^\beta\{\alpha + n(\alpha + m(\beta-1))\} > p$,

(iii) $C \ge 1$ and $m - (p-\alpha)/\beta > k > l > 1$,

where k is 'close enough' to m and n is the largest integer satisfying

$n \le \beta$.

Using (3.3.64) to estimate $|\nabla u|^\beta$ in the right hand side of (3.3.52) we find that inequality (3.3.52) will be satisfied as required if

$$(p-s)\,u^{p+s-1} \ge (\alpha+s(\beta-1))\,u^{\alpha+s-1}\,|Cu^m+u^k-Au^l+B|^\beta + 2\epsilon\gamma(x_1-\delta_0)^{\gamma-1}su^{2s-1} \tag{3.3.65}$$

which we shall call condition E .

As $m - (p-\alpha)/\beta$, however, and we again take $p > s > 1$, it is clear that the highest power of u in the right hand side of inequality (3.3.65) is $p + s - 1$, i.e. the same as the left hand side. It follows, therefore, that condition E will only be satisfied when

$$(p-s) > (\alpha+s(\beta-1))C^{\beta} \tag{3.3.66}$$

(so that this term is of the desired sign) and u is 'large enough' compared to

A, B, C and $\epsilon\gamma$.

We assume for the moment that inequality (3.3.66) is compatible with the other necessary constraints (i.e. the conditions required by the estimate (3.3.64) and labelled (i)-(iii)).

Returning to inequality (3.3.52), we see that this is also satisfied as required if

$$\frac{\gamma(\gamma-1)u^{s}}{(x_1-\delta_0)^2} + s(s-1)u^{s-2}|\nabla u|^2 \geq (\alpha+s(\beta-1))u^{\alpha+s-1}|\nabla u|^{\beta} + 2\epsilon\gamma(x_1-\delta_0)^{\gamma-1}su^{2s-1}$$

which can be rewritten as

$$\begin{aligned}\{(s(s-1))^{1/2}u^{1/2(s-2)}|\nabla u| - \tfrac{1}{2}(s(s-1))^{-1/2}(\alpha+s(\beta-1))u^{\alpha+1/2 s}|\nabla u|^{\beta-1}\}^2 \\ - \tfrac{1}{4}(s(s-1))^{-1}(\alpha+s(\beta-1))^2u^{2\alpha+s}|\nabla u|^{2(\beta-1)} \\ + \frac{\gamma(\gamma-1)u^{s}}{(x_1-\delta_0)^2} \geq 2\epsilon\gamma(x_1-\delta_0)^{\gamma-1}su^{2s-1}.\end{aligned} \tag{3.3.67}$$

Inequality (3.3.67), and hence inequality (3.3.52), will also be satisfied if

$$\frac{\gamma(\gamma-1)u^{s}}{(x_1-\delta_0)^2} \geq \tfrac{1}{4}(s(s-1))^{-1}(\alpha+s(\beta-1))^2u^{2\alpha+s}|\nabla u|^{2(\beta-1)} + 2\epsilon\gamma(x_1-\delta_0)^{\gamma-1}su^{2s-1}. \tag{3.3.68}$$

Now, as $\beta \geq 1$, we may again use the estimate (3.4.43) to substitute for $|\nabla u|$ in (3.3.68), which then becomes

$$\frac{\gamma(\gamma-1)u^s}{(x_1-\delta_0)^2} \geq \tfrac{1}{4}(s(s-1))^{-1}(\alpha+s(\beta-1))^2 u^{2\alpha+s}\left|Cu^m+u^k-Au^l+B\right|^{2(\beta-1)}$$
$$+\ 2\epsilon\gamma(x_1-\delta_0)^{\gamma-1}su^{2s-1}.$$

We call this condition F, which is also dependent on satisfaction of the conditions (i)-(iii) required by the use of the estimate (3.3.64).

If these conditions hold, however, then condition F will be satisfied provided u is small enough compared to γ and $1/\epsilon$, as we already have that $\alpha \geq 0$ and $s > 1$.

We conclude, therefore, that if in addition to $p > s > 1$ and (3.2.4), the conditions (3.3.66) and the conditions (i)-(iii) required by the estimate (3.3.64) hold, then by choosing ϵ and $1/\gamma$ small enough, we can ensure that at least one of the conditions E or F will always be satisfied. This in turn ensures that the inequality (3.3.52) is satisfied as required.

It remains to check that for $p > s > 1$, the inequality (3.3.66) and the conditions labelled (i)-(iii) necessary to allow the use of (3.3.64) are compatible.

This analysis has already been performed in Section 2.4 as part of the proof of Theorem 2.4.3, and the values of α, β and p for which the above conditions hold are summarised as 'Case 3' in the statement of Theorem 2.4.3. In this analysis, however, we are only interested in the question 'does a suitable s such that $p > s > 1$ exist' and not, as in Theorem 2.4.3, in what the applicable value of s actually is. Hence the 'Case 3' option of Theorem 2.4.3 may simplified in this case by noting that there exists some s, with $p > s > 1$, for which condition (3.3.66) and the conditions (i)-(iii) required by the estimate (3.3.64) hold provided

(a) $\beta > 2(p-\alpha)/(p+1)$ and $\alpha + n(\alpha + m(\beta - 1)) \geq p$,

or (b) $\beta > 2(p-\alpha)/(p+1)$, $\alpha + n(\alpha + m(\beta - 1)) < p$, and $p > 1 + \beta/n$,

where $m = (p-\alpha)/\beta$ and n is the largest integer satisfying $n \leq \beta$. Again, as in Theorem 2.4.3, it may be noted that as $p > \alpha + \beta,\ \alpha \geq 0,\ \beta \geq 1$ and $p \geq 1$, the condition $p > 1 + \beta/n$ in (b) is automatic if $\alpha \geq 1,\ \beta \geq 2$, or $p \geq 3$.

In either of the cases 1, 2 or 3 where inequality (3.3.52) has been verified for some $s > 1$, we recall that this allows the conclusion that a positive maximum of the function $J(x,t)$ (as defined in (3.3.40)) is impossible within the interior of the set $\Omega^{+}_{\delta_0} x(0,T)$, provided the constants ϵ and $1/\gamma$ are suitably small.

From (3.3.40), we also see that

$$
\begin{aligned}
J(x,t) &= u_{x_1}(x,t) + \epsilon(x_1-\delta_0)^{\gamma} u^{s} \\
&= u_{x_1}(x,t) \\
&< 0 \quad \text{for} \quad x_1 = \delta_0 \quad \text{by (3.3.39).}
\end{aligned}
$$

Further,

$$
J(x,0) = \varphi_{x_1}(x,t) + \epsilon(x_1-\delta_0)^{\gamma} \varphi^{s}(x)
$$

can be made strictly less than zero throughout $\Omega^{+}_{\delta_0}$ provided ϵ is small

enough, in light of (3.3.27) and as x_1 is taken to be an outward pointing vector

on $\partial\Omega$.

Finally, if Γ is defined as

$$
\Gamma = \Omega^{+}_{\delta_0} \cap \partial\Omega \ ,
$$

then as $u = 0$ for $x \in \partial\Omega$, and as $\dfrac{\partial u}{\partial x_1} < 0$ for $x \in \Omega^{+}_{\delta_0}$ by (3.3.39), it follows

that

$$
\begin{aligned}
J(x,t) &= \frac{\partial u}{\partial x_1}(x,t) + \epsilon(x_1-\delta_0)^{\gamma} u^{s}(x,t) \\
&= \frac{\partial u}{\partial x_1}(x,t) \\
&< 0 \quad \text{for} \quad x \in \Gamma x (0,T) \ .
\end{aligned}
$$

Hence we see that, for small ϵ, $J < 0$ on the parabolic boundary of

$\Omega^{+}_{\delta_0} x (0, T)$. We conclude that

$$J = u_{x_1} + \epsilon (x_1 - \delta_0)^{\gamma} u^{s} < 0 \text{ in } \Omega^{+}_{\delta_0} x (0, T) ,$$

so that

$$-u_{x_1} = | u_{x_1} | > \epsilon (x_1 - \delta_0)^{\gamma} u^{s} \tag{3.3.69}$$

where $x' = (x_2, \ldots x_N) = 0$ and $\delta_0 \le x_1 < 0$.

As $u(x, t) > 0$ within $\Omega^{+}_{\delta_0}$, it follows on integrating (3.3.69) with respect to

x_1 , that for each $y_1 \epsilon \Omega^{+}_{\delta_0}$, i.e. $\delta_0 < y_1 < 0$, then

$$\frac{1}{(s-1) u^{s-1} (y_1, 0, t)} - \frac{1}{(s-1) u^{s-1} (\delta_0, 0, t)} > \frac{\epsilon (y_1 - \delta_0)^{\gamma+1}}{(\gamma+1)} ,$$

so that, as $u(\delta_0, 0, t) > 0$, we find that

$$u(y_1, 0, t) \le \{\epsilon (s - 1) (y_1 - \delta_0)^{\gamma+1} / (\gamma + 1)\}^{-1/(s-1)} . \tag{3.3.70}$$

Hence, if δ_1 is defined as $\delta_1 = \delta_0 - \epsilon_0$ for any $\epsilon_0 > 0$, then for any

$\delta_1 < y_1 < 0$ we apply the estimate (3.3.70) to deduce that

$$u(y_1,0,t) \le \{\varepsilon(s-1)(\delta_1-\delta_0)^{\gamma+1}/(\gamma+1)\}^{-1/(s-1)}$$

for any $\delta_1 < y_1 < 0,\ 0 < t < T$.

It follows, therefore, that every point in the set

$$\{x' = 0,\ \delta_1 - \delta_0 - \epsilon_0 < x_1 < 0\}$$

is not a blow-up point of u for small enough ϵ_0 .

It is also clear that δ_0 (and hence δ_1) may be chosen independently of the initial point $y_0 \in \partial\Omega$. By varying y_0 along $\partial\Omega$ we conclude that there exists a neighbourhood, Ω' , of $\partial\Omega$, such that each point x in Ω' is not a blow-up point.

We may now state the following Theorem.

Theorem 3.3.3

Suppose $\varphi(x)$ is large enough compared to Ω and satisfies condition (3.2.5) and that $\alpha,\ \beta$ and p satisfy (3.2.4) and (3.2.64), then the solution, u , to problem (3.2.1)-(3.2.3) will, according to Theorem 3.2.4, blow-up at some finite time T .

Suppose further that Ω is a convex domain in $\mathbf{R}^N$ and that $\varphi(x)$ also satisfies the condition (3.3.27), then the set of blow-up points is a compact subset of Ω provided α, β and p also satisfy the requirements of one of the cases 1-3 described below.

Case 1 $\quad \beta < 2(p-\alpha)/(p+1)$,

Case 2 $\quad \beta = 2(p-\alpha)/(p+1)$,

Case 3 $\quad \beta > 2(p-\alpha)/(p+1)$ and either

(a) $\quad \alpha + n(\alpha + m(\beta - 1)) \geq p$, or

(b) $\quad \alpha + n(\alpha + m(\beta - 1)) < p$ and $p > 1 + \beta/n$,

where $m = (p-\alpha)/\beta$ and n is the integer part of β .

Remark

It is clear from the alternatives offered in Cases 1-3 above that α, β and p need only satisfy the conditions (3.2.4) and (3.2.64) if $\alpha \geq 1$, $\beta \geq 2$, or $p \geq 3$ (as the requirement $p > 1 + \beta/n$ in Case 3(b) would then be automatic).

Section 3.4 Estimate of blow-up rate for the negative gradient case

Sections 3.2 and 3.3 have considered the questions 'does the solution to (3.2.1)-(3.2.4) exhibit finite-time blow-up', and if so 'where does this blow-up take place'. This work has tried, in general, to extend the techniques developed in Chapter 2 to this more general higher-dimensional problem with varying degrees of success. In this section we investigate the rate at which such a solution may blow up and find that, if the considered function satisfies the requirements of the preceeding subsections, then the full range of one-dimensional results in this area (described in detail in Section 2.4) may be verified in the higher dimensional case.

Throughout this section we consider a solution to problem (3.2.1)-(3.2.4) which satisfies the requirements of Theorem 3.3.3 so that blow-up occurs at some finite-time T within a compact subset of the convex domain Ω .

The techniques used in Section 2.4 to establish Theorem 2.4.1 may be applied directly to this function, and this, along with Corollary 2.4.2, now yields:-

Theorem 3.4.1

If the function $m(t)$ is defined, as in Section 2.4, as

$$m(t) = \max_{x\in\Omega} u(x,t) \ , \tag{3.4.1}$$

then $m(t)$ is Lipschitz continuous, and

$$m'(t) \leq m^p(t) \tag{3.4.2}$$

at any point at which the function m is differentiable.

The proof of this result is identical to that of Theorem 2.4.1 of Section 2.4, or indeed to its predecessor, Theorem 4.5 of Friedman & MacLeod 1985.

Hence, as $m(T) = +\infty$ by assumption, it follows if (3.4.2) is integrated from t to T that there exists a constant $C > 0$ such that

$$m(t) = \max_{x\in\Omega} u(x,t) \geq \frac{C}{(T-t)^{1/(p-1)}} \qquad \text{for any } \; 0 < t < T \,. \tag{3.4.3}$$

We proceed, as in Section 2.4, to try to establish a complimentary estimate, and make the assumption that $\varphi(x)$ can be chosen to satisfy, in addition to (3.2.4) and (3.3.27),

$$\varphi \in C^2(\Omega) \;, \text{ with } \; \nabla^2\varphi + \varphi^p - \varphi^\alpha |\nabla\varphi|^\beta \geq 0 \quad \text{in } \; \Omega \,.$$

This assumption ensures that $u_t(x,0) \geq 0$ in Ω, and, as $u_t = 0$ on $\partial\Omega$ for $0 < t < T$, that u_t is no less than zero on the parabolic boundary of Ω_T.

Further, on differentiating (3.2.1) with respect to t we see that, if $w = u_t$, then w satisfies

$$w_t = \nabla^2 w + p u^{p-1} w - \alpha u^{\alpha-1} |\nabla u|^\beta w - \beta u^\alpha |\nabla u|^{\beta-2} \nabla u . \nabla w$$

$$\text{in } \; \Omega, \; 0 < t < T \,, \tag{3.4.5}$$

so that

$$w_t - \nabla^2 w - pu^{p-1}w + \beta u^{\alpha} |\nabla u|^{\beta-2}\nabla u.\nabla w = -\alpha u^{\alpha-1} |\nabla u|^{\beta} w ,$$

$$\text{in } \Omega,\ 0 < t < T . \qquad (3.4.6)$$

The condition (3.2.4) guarantees that each of the terms on the left hand side of (3.4.6) remain bounded throughout Ω_T . Further, at a non-positive interior minimum of w we would have that

$$w_t - \nabla^2 w - pu^{p-1}w + \beta u^{\alpha} |\nabla u|^{\beta-2}\nabla u.\nabla w = -\alpha u^{\alpha-1} |\nabla u|^{\beta} w \geq 0$$

which would be a contradiction to the maximum principle. It follows, therefore, that a non-positive minimum of w is impossible within the interior of Ω_T , and hence that

$$u_t > 0 \quad \text{for} \quad x\epsilon\Omega,\ 0 < t < T . \qquad (3.4.7)$$

This observation can now be used to derive a 'better' lower bound for u_t away from the parabolic boundary of Ω_T .

For any $\eta > 0$, we define the set Ω^{η} as

$$\Omega^{\eta} = \{x\epsilon\Omega : dist(x,\partial\Omega) > \eta\} , \qquad (3.4.8)$$

where for any point x and any set A ,

$$dist(x,A) = \min_{y\in A} \|x-y\| \quad .$$

We consider the function $J(x,t)$ defined as

$$J(x,t) = u_t(x,t) - \epsilon c(t) g(u(x,t)) \tag{3.4.9}$$

where ϵ is a small positive constant and

$$c(t) = e^{-Mt}, \; g(u) = u^s \tag{3.4.10}$$

for some positive M and $s > 1$.

From Theorem 3.3.3 we see that, if η is sufficiently small, then

$$g(u) = u^s \le C_0 < \infty \text{ if } x\in\partial\Omega^\eta, \; 0 < t < T \; . \tag{3.4.11}$$

Further, from (3.4.7) we see that

$$u_t \ge C_1 > 0 \tag{3.4.12}$$

on the parabolic boundary of the set $\Omega^\eta x(\eta, T)$. It follows, therefore, that if ϵ is small enough, then $J(x,t)$ will be strictly positive on the parabolic boundary of $\Omega^\eta x(\eta, T)$.

Next, on differentiating (3.4.9), we find that

$$J_t = u_{tt} - \epsilon c' g - \epsilon c g' u_t \; , \tag{3.4.13}$$

$$\nabla J = \nabla u_t - \epsilon c g' \nabla u \ , \tag{3.4.14}$$

and $$\nabla^2 J = \nabla^2 u_t - \epsilon c g' \nabla^2 u - \epsilon c g'' |\nabla u|^2 \ , \tag{3.4.15}$$

so that

$$\begin{aligned} J_t - \nabla^2 J = {} & p u^{p-1} u_t - \alpha u^{\alpha-1} |\nabla u|^{\beta} u_t - \beta u^{\alpha} |\nabla u|^{\beta-2} \nabla u . \nabla u_t \\ & - \epsilon c' g + \epsilon c g'' |\nabla u|^2 - \epsilon c g' u^p + \epsilon c g' u^{\alpha} |\nabla u|^{\beta} . \end{aligned} \tag{3.4.16}$$

If we substitute for u_t and ∇u_t in the right hand side of (3.4.16) by using equations (3.4.9) and (3.4.14) we find that

$$J_t - \nabla^2 J + \beta u^{\alpha} |\nabla u|^{\beta-2} \nabla u . \nabla J - (p u^{p-1} - \alpha u^{\alpha-1} |\nabla u|^{\beta}) J = S \tag{3.4.17}$$

where

$$\begin{aligned} S = {} & \epsilon c \{ p u^{p-1} g(u) - u^p g'(u) \} - \alpha u^{\alpha-1} | \nabla u |^{\beta} \epsilon c g(u) \\ & - (\beta - 1) u^{\alpha} | \nabla u |^{\beta} \epsilon c g'(u) - \epsilon c' g(u) + \epsilon c g''(u) | \nabla u |^2 , \end{aligned} \tag{3.4.18}$$

and each of the terms on the left hand side of (3.4.17) are bounded throughout $\Omega^{\eta} x (\eta , T)$.

It follows, therefore, that if S , as defined by (3.4.18), could be shown to be no less than zero throughout $\Omega^{\eta} x (\eta , T)$, then by applying the maximum principle to

(3.4.17), as we have already demonstrated that $J(x,t)$ is strictly greater than zero on the parabolic boundary of this set, we may conclude that $J(x,t)$ is strictly greater than zero throughout $\Omega^{\eta}x(\eta,T)$.

On substituting for $c(t)$ and $g(u)$ by (3.4.10) in (3.4.18) we see that,

$$S = \epsilon e^{-Mt}\{(p-s)u^{p+s-1} - (\alpha+s(\beta-1))u^{\alpha+s-1}|\nabla u|^{\beta} + Mu^{s} + s(s-1)u^{s-2}|\nabla u|^{2}\} \tag{3.4.19}$$

and we wish to show that the right hand side of (3.4.19) is greater than or equal to zero throughout $\Omega^{\eta}x(\eta,T)$.

The verification of this condition makes use of the upper bound for the gradient of the considered solution established in Appendix A. As a result it is straightforward but relatively detailed. It is possible to avoid repeating these details, however, by re-examining the inequality (2.4.20) considered in Section 2.4 in the analogous one-dimensional analysis. If we substitute $|\nabla u|$ for $|u_x|$ in inequality (2.4.20) then it is clear that the right hand side of inequality (2.4.20) is identical to the right hand side of equation (3.4.19). Further, the sign of the right hand side of inequality (2.4.20) is examined in detail in Section 2.4, and the arguments used, being unaffected by the substitution of $|\nabla u|$ for $|u_x|$, would be equally valid if applied to equation (3.4.19). By this route, it is clear that the right hand side of equation (3.4.19) will be greater than or equal to zero as required, in the circumstances summarised in the statement of Theorem 2.4.3 on page 217 and recalled here as follows:-

Case 1 If $\beta < 2(p-\alpha)/(p+1)$, then the right hand side of equation (3.4.19) is greater than or equal to zero for any s such that

$$p \geq s > 1 \ .$$

Case 2 If $\beta = 2(p-\alpha)/(p+1)$, then the right hand side of equation (3.4.19) is greater than or equal to zero for

$$s = (p-\alpha)/\beta = \tfrac{1}{2}(p+1) > 1 \ .$$

Case 3 If $\beta > 2(p-\alpha)/(p+1)$, then the right hand side of equation (3.4.19) is greater than or equal to zero

(A) for $(p-\alpha)/\beta > s > 1$

if $p \leq \alpha + n(\alpha + m(\beta - 1))$ where $m = (p-\alpha)/\beta$ and n is the integer part of β (n is the largest integer satisfying $n \leq \beta$), or

(B) for $\frac{pn}{(n+\beta)} > s > 1$

if $p > \alpha + n(\alpha + m(\beta - 1))$ and $p > 1+\beta/n$.

The alternatives offered in Cases 1-3 above are seen, on comparison, to be identical to those required by Theorem 3.4.1 which we have taken to hold throughout this section. Hence we see that the term s , as described in equation (3.4.19) is greater than or equal to zero as required for some $s > 1$ and all α, β and p considered.

We therefore conclude, as proposed, that $J(x,t)$ as defined by (3.4.9), is strictly greater than zero throughout the set $\Omega^\eta x(\eta,T)$ for s as described in Cases 1-3 above, and provided ϵ, η and $1/M$ are chosen suitably small. Hence,

$$J(x,t) = u_t(x,t) - \epsilon e^{-Mt} u^s(x,t) > 0 \quad \text{in} \quad \Omega^\eta x(\eta,T) \; . \tag{3.4.20}$$

It follows, as $s > 1$ in each of the Cases 1-3 described, and as $u > 0$ in $\Omega^\eta x(\eta,T)$, that if inequality (3.4.20) is integrated from t to T , then there exists a constant $C > 0$ for which

$$u(x,t) \le \frac{C}{(T-t)^{1/(s-1)}} \qquad \text{for} \quad (x,t) \in \Omega^\eta x(\eta,T) \; .$$

Further, if η is small enough, then Theorem 3.3.3 ensures that $u(x,t)$ will also remain bounded at any other points within Ω_T. Hence, there exists a constant, C_1 say, such that

$$u(x,t) \le \frac{C_1}{(T-t)^{1/(s-1)}} \qquad \text{for} \quad (x,t) \in \Omega_T \qquad (3.3.21)$$

and where s takes the particular values described in Cases 1-3. We are now able to formulate the following Theorem.

Theorem 3.4.2

If $u(x,t)$ is a solution to the problem (3.2.1)-(3.2.3) for which Theorem 3.3.3 applies, then there exists a constant C_1 such that inequality (3.4.21) holds. In this inequality, available values of the constant $s > 1$ are determined, in general, by the variation of the parameter β about the value $2(p-\alpha)/(p+1)$. This variation takes exactly the form observed in the proof of Theorem 3.3.3, and particular values of available s are described by the Cases 1-3 of page 302.

Section 3.5 Existence of blow-up for a positive gradient term

The study, in Sections 2.2-2.4, of the one-dimensional problem (2.2.1)-(2.2.3) identified cases where the form chosen for the gradient term u_x^β (i.e. $u_x^\beta = |u_x|^\beta$ or $u_x^\beta = |u_x|^{\beta-1}u_x$), can influence the amount of information available about the size and location of the region within which finite-time blow-up may be expected to occur. This distinction becomes important (in its influence on the positive identification of solutions which exhibit single-point blow-up) in the particular case of $\beta = 1$, as described in Theorem 2.3.7. Although most of the results of these sections are independent of the choice of the form of this term, this observation led to the conclusion that solutions to an equation such as (2.2.1) in which the gradient term were wholly positive throughout the considered region, may, if finite-time blow-up takes place, exhibit interesting blow-up behaviour which need not bear any relation to the results established for the problem (2.2.1)-(2.2.3).

It was anticipated that the techniques developed in Sections 2.2-2.4 would prove to be useful tools if used to investigate the blow-up behaviour of such problems, however, the one-dimensional form of which were subsequently studied in Sections 2.5-2.7.

The results of these sections clearly illustrate that, although the existence of blow-up for the type of equation studied, (i.e. the problem (2.5.1)-(2.5.3)) can be established by a relatively straightforward process, the identification of the appropriate blow-up set is certainly not a simple task. The one-dimensional results in this area do also indicate that a possibly wider range of blow-up behaviour may be observable, even for symmetric solutions to such equations, than for a similar equation of the form studied in Sections 2.2-2.4. Hence, although unable to provide a complete description of where such problems may exhibit finite-time blow-up, similar techniques to those developed in Sections 2.2-2.4 were able to establish single-point blow-up for a large number of equations of this type, and to provide estimates to the rate at which blow-up may take place.

In this section we consider the higher dimensional form of equations of this type and anticipate that similar blow-up behaviour may be observed. It is also anticipated, however, that no more information will be available in the higher dimensional case than was obtainable in the (usually more straightforward) one-dimensional problem. This may be a result of the form of blow-up which <u>does</u> occur, however, rather than any failure on the behalf of the techniques used (i.e. the techniques used seek, where possible, to identify single-point blow-up and consequently yield no firm information if unsuccessful. It is possible, therefore, that for this type of problem, single-point blow-up is not identified simply because blow-up does not take place at a single point.) The object of this section is to investigate blow-up, and the blow-up behaviour of the N-dimensional extension of the problem (2.5.1)-(2.5.3), and where we anticipate that the techniques developed in Section 2.5-2.7, and indeed Sections 3.2-3.4, may prove useful in this task.

Throughout this section, $u(x,t)$ is taken to be a solution to the problem

$$u_t = \nabla^2 u + u^p + u^\alpha |\nabla u|^\beta \qquad \text{in } \Omega,\ t > 0 \tag{3.5.1}$$

$$u(x,0) = \varphi(x) \qquad \text{in } \Omega \tag{3.5.2}$$

$$u(x,t) = 0 \qquad \text{on } \partial\Omega,\ t > 0\ , \tag{3.5.3}$$

where Ω is a bounded region in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, and where, minimally,

$$p > 1, \ \alpha \geq 0 \ \text{ and } \ \beta \geq 0 \ . \tag{3.5.4}$$

It is also assumed that $\varphi(x)$ satisfies the condition (3.2.4) of Section 3.2 and re-introduced as

$$\varphi \in C^0(\Omega), \ \varphi \geq 0 \ \text{ and } \ \varphi(x) = 0 \ \text{ for } \ x \ \text{ on } \ \partial\Omega \ . \tag{3.5.5}$$

In order to establish blow-up for u , we again make use of the results of Lacey 1983, which, through not being restricted to the one-dimensional problem, allow the proof of Theorem 2.5.1 from Section 2.5 to be applied directly to the problem (3.5.1)-(3.5.4). In the higher dimensional case this result becomes

Theorem 3.5.1

If $\varphi(x)$ is 'large enough' compared to Ω and satisfies (3.5.5), then the solution, $u(x,t)$, to the problem (3.5.1)-(3.5.3) with α, β and p satisfying (3.5.4) will blow-up at a finite time, say T .

Proof

The proof of this result is identified to that of Theorem 2.5.1 and is not repeated. As in Section 2.5, Theorem 3.5.1 demonstrates that finite-time blow-up of the solution to problem (3.5.1)-(3.5.4) can occur for a large class of initial conditions φ . We continue by investigating where this blow-up may take place, and begin by considering the radially symmetric problem.

Section 3.6 Identification of the blow-up sets

3.6.1 The symmetric problem

In this subsection, we consider the problem (3.5.1)-(3.5.4) in the case that

$$\Omega \text{ is in the ball, } B_R = \{\, |x| < R \,\} \subset \mathbf{R}^N \tag{3.6.1}$$

and where, in addition to (3.5.5), the initial condition $\varphi(x)$ satisfies

$$\varphi = \varphi(r)\ , \quad \varphi'(r) < 0 \quad \text{for} \quad 0 < r = |x| \le R\ ,$$

$$\text{and} \quad \varphi''(0) < 0\ . \tag{3.6.2}$$

We also take φ large enough to make Theorem 3.5.1 applicable and denote the finite blow-up time of u by T.

In light of the assumptions (3.6.1) and (3.6.2) it follows that u is radial, i.e.

$$u = u(r,t)\ . \tag{3.6.3}$$

In this case, the problem (3.5.1)-(3.5.3) can be written as,

$$u_t = u_{rr} + \frac{(N-1)}{r} u_r + u^p + u^{\alpha} |u_r|^{\beta}\ , \quad \text{in } B_R,\ t > 0 \tag{3.6.4}$$

$$u(r,0) = \varphi(r)\ , \qquad \text{in } B_R\ , \tag{3.6.5}$$

$$u(R,t) = 0\ , \qquad \text{for } t > 0\ , \tag{3.6.6}$$

and again,

$$p > 1,\ \alpha \geq 0 \text{ and } \beta \geq 0\ . \tag{3.5.4}$$

We begin by establishing the following Lemma.

Lemma 3.6.1

If B_T denotes $B_R \times (0,T)$, and $u(r,t)$ is a solution to (3.6.4)-(3.6.6) with

α, β and p as described in (3.5.4) and $\varphi(x)$ satisfying (3.5.5) and (3.6.2),

then

$$u_r < 0 \qquad \text{in } B_T \cap \{r > 0\}\ .$$

Proof

The proof of this result follows, exactly, the technique used in Section 3.3 to establish Lemma 3.3.1. If the function $w(r,t)$ is defined as

$$w(r,t) = r^{N-1}u_r(r,t)\ , \tag{3.6.7}$$

then on differentiating,

$$\begin{aligned} w_t - w_{rr} + \frac{(N-1)}{r}w_r - pu^{p-1}w - \frac{\beta u^{\alpha}|w|^{\beta-1}sign(w)\,w_r}{r^{(\beta-1)(N-1)}} \\ + \frac{(N-1)\beta u^{\alpha}|w|^{\beta}}{r^{(\beta-1)(N-1)+1}} - \frac{\alpha u^{\alpha-1}|w|^{\beta}w}{r^{\beta(N-1)}} = 0. \end{aligned} \tag{3.6.8}$$

As $N > 1$,

$$w(r,t) = r^{N-1}u_r(r,t) = 0 \qquad \text{at } r=0\ , \tag{3.6.9}$$

and a positive maximum of w is clearly impossible at this point.

Further, the maximum principle applied to (3.6.4)-(3.6.6) yields that

$$u > 0 \quad \text{within} \quad B_T \,, \tag{3.6.10}$$

and hence also that $u > 0$ at any interior maximum of w.

It follows, therefore, that each of the terms in equation (3.6.8) must be bounded at any interior maximum of w, which allows the maximum principle to be applied to establish that a positive interior maximum of w is impossible within B_T.

In addition, (3.6.6) and (3.6.10) show that

$$w = r^{N-1}u_r < 0 \qquad \text{on} \quad r = R,\ t > 0 \,,$$

and from (3.6.2)

$$w = r^{N-1}\varphi' < 0 \qquad \text{at} \quad t = 0,\ 0 < r \le R \,.$$

Hence, we see that a positive maximum of w also cannot occur on the parabolic boundary of B_T which establishes Lemma 3.6.1.

With Lemma 3.6.1, it can be shown that, under appropriate conditions on the parameters α, β and p, the solution to (3.6.4)-(3.6.6) will blow-up at a single point.

This analysis is similar to that of Section 3.3 and considers the sign of $J(r,t)$,

$$J(r,t) = w(r,t) + \epsilon c(r)\, g(u(r,t)) \,, \quad \text{in} \quad B_T \,, \tag{3.6.11}$$

where $\epsilon > 0$ is small with $c(r)$ and $g(u)$ positive and satisfying

$$c(0) = g(0) = 0 \ . \tag{3.6.12}$$

In the following analysis we show that, if ϵ is sufficiently small, and if α, β and p are suitably related, then J will be no greater than zero throughout B_T provided the functions c and g are chosen appropriately.

On the parabolic boundary of B_T we see that

$$\begin{aligned} J(R,t) &= w(R,t) + \epsilon c(R) g(0) \\ &= w(R,t) \\ &\le 0 \quad \text{by (3.6.12) and Lemma 3.6.1,} \end{aligned} \tag{3.6.13}$$

$$\begin{aligned} J(0,t) &= w(0,t) + \epsilon c(0) g(u) \\ &= w(0,t) \\ &= 0 \quad \text{by (3.6.12) and (3.6.6),} \end{aligned} \tag{3.6.14}$$

and

$$\begin{aligned} J(r,0) &= r^{N-1}\varphi'(r) + \epsilon c(r) g(\varphi) \\ &\le 0 \quad \text{for} \quad 0 \le r < R \text{, by (3.6.2),} \end{aligned} \tag{3.6.15}$$

if ϵ is small enough and we choose

$$c(r) \le r^N \qquad \text{for small} \quad r \ . \tag{3.6.16}$$

Hence $J \le 0$ on the parabolic boundary of B_T .

As in Section 3.3, we differentiate (3.6.11) and in this case see that

$$J_t - J_{rr} = -\frac{(N-1)}{r} w_r + pu^{p-1}w - \frac{\beta u^\alpha |w|^{\beta-1} w_r}{r^{(\beta-1)(N-1)}}$$
$$-\frac{(N-1)\beta u^\alpha |w|^\beta}{r^{(N-1)(\beta-1)+1}} - \frac{\alpha u^{\alpha-1}|w|^{\beta+1}}{r^{\beta(N-1)}} + \frac{\epsilon c g'(N-1) w}{r^N}$$
$$+\epsilon c g' u^p + \frac{\epsilon c g' u^\alpha |w|^\beta}{r^{\beta(N-1)}} - \epsilon c'' g - \frac{2\epsilon c' g' w}{r^{(N-1)}} - \frac{\epsilon c g'' w^2}{r^{2(N-1)}}. \tag{3.6.17}$$

From (3.6.14) we see that $J = 0$ at $r = 0$ and from Lemma 3.6.1, (3.6.11) and (3.6.12), that $J \leq 0$ at any point at which $u = 0$. Hence, neither r nor u can be zero at a positive maximum of J and the terms in equation (3.6.17) must be bounded at any such point.

We next assume $g'' \geq 0$ and substitute for w_r and w (an expression for w_r is obtained by differentiating (3.6.11)) in (3.6.17) to see that

$$J_t - J_{rr} + aJ_r + bJ \leq S\ . \tag{3.6.18}$$

In this inequality, the functions a and b are necessarily bounded at any positive maximum of J and

$$S = \epsilon(N-1)c'g + \frac{2\epsilon^2 c g g'}{r^{(N-1)}}\left\{c' - \frac{c(N-1)}{r}\right\}$$
$$-\epsilon c\left[pu^{p-1}g - g'u^p\right] - \frac{\epsilon^{\beta+1}c^{\beta+1}g^\beta u^{\alpha-1}}{r^{\beta(N-1)}}\{\alpha g + (\beta-1)ug'\} \tag{3.6.19}$$
$$+\frac{\beta\epsilon^\beta c^{\beta-1}u^\alpha g^\beta}{r^{(N-1)(\beta-1)}}\left\{c' - \frac{c(N-1)}{r}\right\} - \epsilon c'' g.$$

Hence, if we can show that S is no greater than zero throughout B_T, then by the maximum principle applied to (3.6.18) we may conclude that J cannot have a positive maximum within the interior of B_T.

If we set

$$c(r) = r^m \quad \text{and} \quad g(u) = u^k \tag{3.6.20}$$

where $m \geq N$ and $k > 1$ (so that the conditions (3.6.12) and (3.6.16) are satisfied, and $g'' \geq 0$ as required) then (3.6.19) becomes

$$\begin{aligned} S = \epsilon r^m \Big\{ - \frac{m(m-N)\,u^k}{r^2} &+ 2(m-N+1)\,\epsilon k u^{2k-1} r^{(m-N)} \\ &- (p-k)\,u^{p+k-1} - \epsilon^\beta(\alpha+k(\beta-1))\,r^{\beta(m+N-1)} u^{\alpha-1+k(\beta+1)} \\ &+ \beta\epsilon^{\beta-1}(m-N+1)\,r^{(m-N)(\beta-1)+\beta-2} u^{\alpha+k\beta} \Big\}. \end{aligned}$$

If we now make the substitution

$$n = m + N - 1 \tag{3.6.21}$$

we see that S will be less than or equal to zero if

$$\begin{aligned} \frac{(N-1)(n-1)\,u^k}{r^2} + \epsilon^\beta(\alpha+k(\beta-1))\,r^{n\beta} u^{\alpha-1+k(\beta+1)} + \frac{n(n-1)\,u^k}{r^2} \\ + (p-k)\,u^{p+k-1} \geq n\beta\epsilon^{\beta-1} r^{n(\beta-1)-1} u^{\alpha+k\beta} + 2\epsilon n k r^{n-1} u^{2k-1}. \end{aligned} \tag{3.6.22}$$

Hence, as $N \geq 1$, inequality (3.6.22) will be satisfied as required if

$$n \geq 1 \text{ , and } \alpha + k(\beta-1) \geq 0 \text{ ,} \tag{3.6.23}$$

and if

$$\frac{n(n-1)u^k}{r^2} + (p-k)u^{p+k-1} \geq n\beta\epsilon^{\beta-1}r^{n(\beta-1)-1}u^{\alpha+k\beta} + 2\epsilon nkr^{n-1}u^{2k-1} \text{ .} \tag{3.6.24}$$

A range of values of the parameters α, β and p for which inequality (3.6.24) will be satisfied can now be established by making reference to Section 2.3.2. On comparison, inequality (3.6.24) is seen to be identical to inequality (2.3.61) (with r replacing $(x-\gamma)$) considered in Section 2.3.2. Further, the proof of Lemma 2.3.5 shows that inequality (2.3.61) is satisfied for some $k > 1$ if

$$\beta > 1 \text{ and } p > \alpha + \beta \text{ , or if } \beta = 1 \text{ and } p > 2\alpha + 1$$

and provided ϵ and $1/n$ are sufficiently small. This analysis is directly applicable to inequality (3.6.24), however, and allows the identical conclusion, that (3.6.24) is satisfied for all $(x,t) \in B_T$ provided ϵ is small, $n = m - N + 1$ is large, and the condition (3.6.25) holds. We also note that the additional condition (3.6.23) is satisfied by 'large' n and α, β and p satisfying (3.6.25) and this allows the condition that s is less than or equal to zero as required in the circumstances described above. Finally, n will be as large as required if we choose m large enough and we see from the maximum principle applied to

(3.6.18) that J cannot have a positive maximum within B_T if ϵ and $1/m$ are small enough, and if $\alpha,\ \beta$ and p satisfy (in addition to (3.5.4)), the conditions described in (3.6.25).

In addition, (3.6.13)-(3.6.15) show that J is less than or equal to zero on the parabolic boundary of B_T and we conclude that

$$J(r,t) = w(r,t) + \epsilon r^m u^k \le 0 \quad \text{in } B_T\ , \tag{3.6.26}$$

for some $k > 1$.

Hence,

Theorem 3.6.2

Suppose u is a solution to problem (3.5.1)-(3.5.3) in the region

$\Omega = B_R = \{ x\colon |x| < R \}$ with $\alpha,\ \beta$ and p satisfying (3.5.4) and that $\varphi(x)$ satisfies (3.5.5) and (3.6.2) and is large enough to make Theorem 3.5.1 applicable (so that finite-time blow-up occurs at some time T).

Then the blow-up occurs at the single point $x = 0$ if additionally,

$$\beta > 1 \text{ and } p > \alpha + \beta \text{ , or } \beta = 1 \text{ and } p > 2\alpha + 1\ .$$

Proof

For any $\alpha,\ \beta$ and p for which inequality (3.6.26) holds it follows that

$$-w = r^{N-1}u_r \ge \epsilon r^m u^k \qquad \text{in } B_T\ . \tag{3.6.27}$$

Hence as $m > N$ (m is 'large'), $k > 1$ and $u(0,t) > 0$ we may integrate (3.6.27) from 0 to r for any $0 < r < R$ to see that there exists a constant $c > 0$ for which

$$\frac{1}{(k-1)\, u^{k-1}(r,t)} \geq \epsilon c r^{m+2-N} . \tag{3.6.28}$$

If there exists some $r > 0$ for which $u(r,t) \to \infty$ as $t \to T$, then this would contradict (3.6.28). It follows that $r = 0$ is the only possible blow-up point, and, as blow-up occurs, that it does so at the point $r = |x| = 0$.

We note finally that the concluding condition appearing in the statement of Theorem 3.6.2 simply ensures that the necessary inequality (3.6.26) holds.

3.6.2 The non-symmetric problem

In this section we consider a more general form of the problem (3.5.1)-(3.5.3) although we restrict ourselves to the case in which

$$\Omega \text{ is a convex domain in } \mathbf{R}^N . \tag{3.6.29}$$

We again assume that the condition (3.5.4) is satisfied and that the initial condition $\varphi(x)$ satisfies (3.5.5) and is large enough to make Theorem 3.5.1 valid. Hence the considered solution blows up at a finite time which we denote by T .

In more than one-dimension, Section 3.3 has illustrated that the determination of information about the blow-up set is not straightforward in the non-radially symmetric case. The form of equation studied in this section, however, suggests that a variety of blow-up behaviour may be observable in its solutions. Hence, any indication of a particular form of blow-up taking place, even if it comes by way of positively ruling out some alternative behaviour, can be valuable in describing the nature of solutions to problems of this type.

In this section we shall show that for some α, β and p we may again establish an analogue of Theorem 3.3 from Friedman & MacLeod 1985 which will show that blow-up occurs within a compact subset of the domain Ω .

As in Section 3.3 we assume that

$$\frac{\partial\varphi}{\partial\nu} < 0 \qquad \text{on } \partial\Omega \tag{3.6.30}$$

for ν any outward pointing normal to $\partial\Omega$, and consider any $y_0 \in \partial\Omega$. For simplicity, we set $y_0 = 0$ and take the half space $\{x_1 > 0\}$ to be tangent to Ω at y_0 with $x_1 < 0$ for any $x = (x_1,\ x_2,\ \ldots\ x_N) \in \Omega$.

For some $\delta < 0$ with $|\delta|$ small we define the set Ω_δ^+ and Ω_δ^- as

$$\Omega_\delta^+ = \Omega \cap \{x_1 > \delta\} \tag{3.6.31}$$

and

$$\Omega_\delta^- = \{(x_1,\ x') : (2\delta - x_1,\ x') \in \Omega_\delta^+ \} \tag{3.6.32}$$

where $x' = (x_2,\ \ldots\ x_N)$. Hence Ω_δ^- is the reflection of Ω_δ^+ with respect to the plane $\{x_1 = \delta\}$.

If the function $w(x, t)$ is defined as

$$w(x, t) = u(x_1, x', t) - v(x_1, x', t) \qquad \text{for } x \in \Omega_\delta^-,\ 0 < t < T \tag{3.6.33}$$

where

$$v(x_1, x', t) = u(2\delta - x_1, x', t) \tag{3.6.34}$$

then we can use w to show that u is a decreasing function of x_1 within Ω_δ^+ for some δ .

To see this, we differentiate (3.6.33) and find that

$$w_t = \nabla^2 w + c(u,v)\,w + d(u,v)\,|\nabla u|^{\beta} w + v^{\alpha}\{|\nabla u|^{\beta} - |\nabla v|^{\beta}\} \tag{3.6.35}$$

where $c(u,v) = (u^p - v^p)/(u-v)$ and $d(u,v) = (u^\alpha - v^\alpha)/(u-v)$.

If w were to attain a negative minimum at some point within $\Omega_\delta^- \times (0,T)$, then

$|\nabla u|^{\beta} = |\nabla v|^{\beta}$ at this point. Further, if $c(u,v)$ and $d(u,v)$ are bounded at this point then the maximum principle applied to (3.6.65) would lead to a contradiction. It follows that a negative minimum of w can only occur within

$\Omega_\delta^- \times (0,T)$ at some point where either $c(u,v)$ or $d(u,v)$ is unbounded.

The function $c(u,v)$ must remain bounded, however, because $p > 1$ (from (3.5.4)), and $d(u,v)$ can only become unbounded (if $0 < \alpha < 1$) where

$u = v$.

But, if at any $(\bar{x},\bar{t}) \in \Omega_\delta^- \times (0,T)$ we have

$$u(\bar{x},\bar{t}) = v(\bar{x},\bar{t}) \ ,$$

then $w(\bar{x},\bar{t}) = 0$ and $(\bar{x},\bar{t})$ is clearly not a negative minimum of w .

Hence, from the maximum principle applied to (3.6.65) we see that a negative minimum of w is impossible within $\Omega_\delta^- \times (0,T)$.

In addition, from (3.6.33),

$$
\begin{aligned}
w(x,t) &= u(x_1,x',t) - u(2\delta-x_1,x',t)\\
&= u(\delta,x',t) - u(\delta,x',t)\\
&= 0 \quad \text{on} \quad \{x_1 = \delta\}
\end{aligned}
\tag{3.6.36}
$$

and

$$
w(x,t) = u(x_1,x',t) > 0 \qquad \text{on} \quad (\partial\Omega_\delta^- \cap \{x_1 < \delta\}) \times (0,T) \tag{3.6.37}
$$

(because $v(x,t) = 0$ on $\partial\Omega_\delta^- \cap \{x_1 < \delta\}$).

Also, from (3.6.30), it follows that

$$
w(x,0) > 0 \quad \text{for} \quad x \in \Omega_\delta^- \tag{3.6.38}
$$

provided $|\delta|$ is sufficiently small.

We conclude, therefore, that w is no less than zero on the parabolic boundary of $\Omega_\delta^- \times (0,T)$ and that

$$
w > 0 \quad \text{in} \quad \Omega_\delta^- \times (0,T) \;. \tag{3.6.39}
$$

As $w > 0$ within $\Omega_\delta^- \times (0,T)$ with $w = 0$ on $\{x_1 = \delta\}$ and as x_1 is outward pointing from Ω_δ^- at $\{x_1 = \delta\}$ it follows that

$$0 > \frac{\partial w}{\partial x_1} - 2\frac{\partial u}{\partial x_1} \qquad \text{on} \quad \{x_1 = \delta\}\ . \tag{3.6.40}$$

Finally, as δ is arbitrary, we see by varying δ that there exists some $\delta_0 < 0$ for which

$$\frac{\partial u}{\partial x_1} < 0 \qquad \text{for} \quad x \in \Omega^+_{\delta_0},\ 0<t<T\ , \tag{3.6.41}$$

if $|\delta_0|$ is sufficiently small.

We next consider the function $J(x,t)$, where

$$J(x,t) = u_{x_1}(x,t) + \epsilon c(x_1)\, g(u(x,t)) \quad \text{in} \quad \Omega^+_{\delta_0} x (0,T)\ , \tag{3.6.42}$$

for some small $\epsilon > 0$, and positive functions c and g . On differentiating (3.6.42) we can see that

$$\begin{aligned} J_t - \nabla^2 J - \beta u^\alpha |\nabla u|^{\beta-2}\nabla u . \nabla J \\ = p u^{p-1} u_{x_1} + \alpha u^{\alpha-1} |\nabla u|^\beta u_{x_1} - \epsilon\beta u^\alpha |\nabla u|^{\beta-2} c' g u_{x_1} \\ - \epsilon(\beta-1) u^\alpha |\nabla u|^\beta c g' + \epsilon c g' u^p - \epsilon c'' g \\ - 2\epsilon c' g' u_{x_1} - \epsilon c g'' |\nabla u|^2 . \end{aligned} \tag{3.6.43}$$

The intention in this work is to develop an equation, involving J , to which the maximum principle may be applied, and the sign of J established. The form of the right hand side of equation (3.6.43), however, dictates that two separate cases and two separate equations be considered, and where these cases are distinguished by the variation of the parameter β about the value 2.

If $\beta \geq 2$, then it is sufficient to use (3.6.42) to substitute for u_{x_1} in the right hand side of (3.6.43) and we find that

$$J_t - \nabla^2 J - \beta u^\alpha |\nabla u|^{\beta-2} \nabla u . \nabla J + d_1 J = S_1 \tag{3.6.44}$$

where

$$\begin{aligned} S_1 = & - \epsilon c \left[p u^{p-1} g - g' u^p \right] - \epsilon c u^{\alpha-1} |\nabla u|^\beta [\alpha g + (\beta - 1) u g'] \\ & + \epsilon^2 c c' \beta u^\alpha |\nabla u|^{\beta-2} g^2 - \epsilon c'' g + 2\epsilon^2 c c' g g' - \epsilon c g'' |\nabla u|^2 . \end{aligned} \tag{3.6.45}$$

If, however, $\beta < 2$, then provided

$$c'(x_1) \geq 0 \quad \text{throughout} \quad \Omega^+_{\delta_0} \tag{3.6.46}$$

we may estimate

$$\epsilon \beta u^\alpha |\nabla u|^{\beta-2} c' g u_{x_1} \geq \epsilon \beta u^\alpha |u_{x_1}|^{\beta-2} c' g u_{x_1} \tag{3.6.47}$$

because $u_{x_1} < 0$ throughout $\Omega^+_{\delta_0} x (0, T)$ by (3.6.41).

If, when $\beta < 2$, (3.6.47) is used in the right hand side of (3.6.43) and (3.6.42) is then used to substitute for u_{x_1}, we find that

$$J_t - \nabla^2 J - \beta u^\alpha |\nabla u|^{\beta-2} \nabla u . \nabla J + d_2 J \leq S_2 \tag{3.6.48}$$

where

$$S_2 = -\,ec\,[pu^{p-1}g - g'u^{p}] - ecu^{\alpha-1}\,|\nabla u|^{\beta}\,[\alpha g + (\beta-1)\,ug'] \\ + e^{\beta}\beta c^{\beta-1}c'g^{\beta}u^{\alpha} - ec''g + 2e^{2}cc'gg' - ecg''\,|\nabla u|^{2}. \tag{3.6.49}$$

The intention is that the maximum principle be applied to both equations (3.6.44) and (3.6.48) so that the sign of the function J can be determined. Before continuing, however, we note that the condition (3.5.4) alone does not guarantee that the functions d_1 and d_2 and the coefficient of ∇J appearing in (3.6.44) and (3.6.48) remain bounded throughout $\Omega^{+}_{\delta_0}x(0,T)$ as is required. Specifically, the possibility exists that these terms may become unbounded at points where $u = 0$, or, in the case that $\beta < 2$, where $|\nabla u| = 0$.

From (3.6.41), however, it is clear that

$$u_{x_1} < 0 \qquad \text{within} \quad \Omega^{+}_{\delta_0}x(0,T)$$

and hence that $|\nabla u| = 0$ is impossible within this set.

Further, if we assume in addition that

$$g(0) = 0 \tag{3.6.50}$$

then at any point within $\Omega^{+}_{\delta_0}x(0,T)$ at which $u = 0$ (if such a point even exists) we would have

$$
\begin{aligned}
J(x,t) &= u_{x_1}(x,t) + ec(x_1)g(0) \\
&= u_{x_1}(x,t) \\
&< 0 \quad \text{by (3.6.41).}
\end{aligned}
$$

Hence, $u = 0$ is impossible at any point within $\Omega^+_{\delta_0} \times (0,T)$ at which J is greater than or equal to zero.

It follows, therefore, that the coefficients on the left hand sides of the equations (3.6.44) and (3.6.48) will be bounded at any points within $\Omega^+_{\delta_0} \times (0,T)$ at which J is greater than or equal to zero, and certainly at any positive interior maximum of J.

We next choose the functions $c(x_1)$ and $g(u)$ as

$$c(x_1) = (x_1 - \delta_0)^{\gamma} \quad \text{and} \quad g(u) = u^{s} \tag{3.6.51}$$

for some positive γ and s and this choice satisfies (3.6.46) and (3.6.50) as required.

Substituting for $c(x_1)$ and $g(u)$ in both (3.6.45) and (3.6.49) we see that

$$S_1 = -\epsilon c\,\{(p-s)\,u^{p+s-1} + (\alpha+s(\beta-1))\,u^{\alpha+s-1}\,|\nabla u|^{\beta} - \epsilon\gamma\,(x_1-\delta_0)^{\gamma-1}\beta\,u^{\alpha+2s}\,|\nabla u|^{\beta-2} + \frac{\gamma(\gamma-1)\,u^{s}}{(x_1-\delta_0)^2} - 2\epsilon\gamma\,(x_1-\delta_0)^{\gamma-1}s u^{2s-1} + s(s-1)\,u^{s-2}\,|\nabla u|^2\}$$

(3.6.52)

and

$$S_2 = -\epsilon c\,\{(p-s)\,u^{p+s-1} + (\alpha+s(\beta-1))\,u^{\alpha+s-1}\,|\nabla u|^{\beta} - \epsilon^{\beta-1}\beta\gamma\,(x_1-\delta_0)^{\gamma(\beta-1)-1}u^{\alpha+s\beta} + \frac{\gamma(\gamma-1)\,u^{s}}{(x_1-\delta_0)^2} - 2\epsilon\gamma\,(x_1-\delta_0)^{\gamma-1}s u^{2s-1} + s(s-1)\,u^{s-2}\,|\nabla u|^2\}.$$

(3.6.53)

If it can be established that S_1 (as described by (3.6.52)) and S_2 (as described by (3.6.53)), are less than or equal to zero throughout $\Omega^{+}_{\delta_0}x(0,T)$, then by applying the maximum principle to (3.6.44) and (3.6.48) respectively, we may conclude that a positive maximum of J is impossible within this set in either case $\beta \geq 2$ or $\beta < 2$ along with any additional conditions we may require (we have already that $\beta \geq 0$ from (3.5.4)).

We first consider S_1 and see from (3.6.52) that S_1 will be less than or equal to zero (as $c(x_1) \geq 0$ for $x\epsilon\Omega^{+}_{\delta_0}$) if

$$(p-s)\,u^{p+s-1} + (\alpha+s(\beta-1))\,u^{\alpha+s-1}\,|\nabla u|^{\beta} + \frac{\gamma(\gamma-1)\,u^{s}}{(x_1-\delta_0)^2} + s(s-1)\,u^{s-2}\,|\nabla u|^2 \geq \epsilon\gamma\,(x_1-\delta_0)^{\gamma-1}\beta\,u^{\alpha+2s}\,|\nabla u|^{\beta-2} + 2\epsilon\gamma\,(x_1-\delta_0)^{\gamma-1}s u^{2s-1}.$$

(3.6.54)

We next introduce the following inequality which has already been considered in the course of previous work:-

$$\tfrac{1}{2}(p-s)\,u^{p+s-1} + \frac{\tfrac{1}{2}\gamma(\gamma-1)\,u^{s}}{(x_1-\delta_0)^2} \geq 2\epsilon\gamma\,(x_1-\delta_0)^{\gamma-1} s u^{2s-1} \tag{3.6.55}$$

provided ϵ and $1/\gamma$ are sufficiently small and

$$p > s > 1\ . \tag{3.6.56}$$

The inequality (3.6.55) appears, for example, in the proof of Lemma 2.3.5 from Section 2.3.2 (the condition (3.6.56) ensures that largest term is on the left hand side of (3.6.55) for both 'large' and 'small' values of u, and by choosing γ large and ϵ small we can ensure these regions overlap).

With (3.6.55), inequality (3.6.54) requires that

$$\tfrac{1}{2}(p-s)\,u^{p+s-1} + (\alpha+s(\beta-1))\,u^{\alpha+s-1}\,|\nabla u|^{\beta} + \frac{\gamma(\gamma-1)\,u^{s}}{2\,(x_1-\delta_0)^2} + s(s-1)\,u^{s-2}\,|\nabla u|^{2}$$
$$\geq \epsilon\gamma\,(x_1-\delta_0)^{\gamma-1}\beta u^{\alpha+2s}\,|\nabla u|^{\beta-2}. \tag{3.6.57}$$

As we only consider S_1 in the case $\beta \geq 2$ (and as $\alpha \geq 0$ from (3.5.4) and $p > s > 1$ from (3.6.56)) it is clear that

$$(\alpha+s(\beta-1)) > 0$$

and (3.6.57) will be satisfied if

$$(\alpha+s(\beta-1))\,u^{u+s-1}\,|\nabla u|^{\beta} \geq \epsilon\gamma\,(x_1-\delta_0)^{\gamma-1}\beta u^{\alpha+2s}\,|\nabla u|^{\beta-2}\ .$$

If this inequality fails, however, then

$$|\nabla u|^2 \le \frac{\epsilon\gamma(x_1-\delta_0)^{\gamma-1}\beta u^{s+1}}{(\alpha+s(\beta-1))}, \tag{3.6.58}$$

and as $\beta \ge 2$, we may use (3.6.58) to estimate $|\nabla u|^{\beta-2}$ in the right hand side of (3.6.57). Hence in this case, inequality (3.6.57) will be satisfied as required if

$$\begin{aligned}\tfrac{1}{2}(p-s)u^{p+s-1} + (\alpha+s(\beta-1))u^{\alpha+s-1}|\nabla u|^\beta + \frac{\gamma(\gamma-1)u^s}{2(x_1-\delta_0)^2} \\ + s(s-1)u^{s-2}|\nabla u|^2 \ge Du^{\alpha+2s+\frac{1}{2}(s+1)(\beta-2)},\end{aligned} \tag{3.6.59}$$

where

$$D = \frac{\{\epsilon\gamma(x_1-\delta_0)^{\gamma-1}\beta\}^{1+\frac{1}{2}(\beta-2)}}{(\alpha+s(\beta-1))^{\frac{1}{2}(\beta-2)}}. \tag{3.6.60}$$

Further, because $\beta \ge 2$, the constant D may be forced to be as small as required by suitable choice of (small enough) ϵ.

To continue, we see that inequality (3.6.59) will be satisfied as required if

$$\tfrac{1}{2}(p-s)u^{p+s-1} \ge Du^{\alpha+2s+\frac{1}{2}(s+1)(\beta-2)}$$

which we call condition A, or if

$$\frac{\frac{1}{2}\gamma(\gamma-1)u^s}{(x_1-\delta_0)^2} \ge Du^{\alpha+2s+\frac{1}{2}(s+1)(\beta-2)}$$

which we call condition B.

Condition A is satisfied if

$$p+s-1 \ge \alpha+2s+\tfrac{1}{2}(s+1)(\beta-2)$$

i.e. if

$$2(p-\alpha)/\beta \geq s + 1 \tag{3.6.61}$$

and if u is large enough compared to D, or if D is 'small enough' in the case of equality in (3.6.61).

Alternatively, condition B will be satisfied if

$$s < \alpha + 2s + \tfrac{1}{2}(s + 1)(\beta - 2) \ ,$$

(which is automatic as $\alpha \geq 0,\ s > 1$ and $\beta \geq 2$), and if u is 'small enough' compared to $1/D$.

Hence, because we can make D as small as required if we choose ϵ small enough, we see that, if the condition (3.6.61) is satisfied in addition to (3.6.56), then at least one of conditions A or B will always be true. It follows, therefore, that inequality (3.6.54) is satisfied and that S_1 is less than or equal to zero if in addition to (3.5.4), and $\beta \geq 2$, the conditions (3.6.56) and (3.6.61) hold and if both ϵ and $1/\gamma$ are chosen suitably small.

We next consider the term S_2 (as described by (3.6.53)) and try to show that this is also less than or equal to zero throughout $\Omega^{+}_{\delta_0}x(0,T)$.

If we assume that

$$\alpha + s(\beta-1) \geq 0 \quad \text{and} \quad s \geq 1 \tag{3.6.62}$$

then the sign of s_2 can be determined by arguments identical to those used in the study of the one-dimensional problem, and also recalled in Section 3.6.1. The conditions (3.6.62) allow us to neglect (two) negative terms from the left hand side of (3.6.53) and (as $c(x_1) \geq 0$ throughout $\Omega^{+}_{\delta_0}$) we see that s_2 will be less than or equal to zero as required if

$$(p-s)\,u^{p+s-1} + \frac{\gamma(\gamma-1)\,u^{s}}{(x_1-\delta_0)^2} \geq \gamma\beta e^{\beta-1}(x_1-\delta_0)^{\gamma(\beta-1)-1}u^{\alpha+s\beta} + 2\epsilon\gamma s(x_1-\delta_0)^{\gamma-1}u^{2s-1}. \tag{3.6.63}$$

The inequality (3.6.63) has, however, already been discussed in Section 2.3.2 (as inequality (2.3.61) with $\gamma - n$, $s - k$ and $\delta_0 - \gamma$) and again in Section 3.6.1 (as inequality (3.6.24) with $\gamma - n$, $s - k$ and $(x_1-\delta_0) - r$).

With reference to previous work, therefore, we again conclude that inequality (3.6.63) will be satisfied for some $s > 1$ (details of allowable s are available although we only require the existence of some $s > 1$ here) if

$$\beta > 1 \text{ and } p > \alpha + \beta\text{, or if } \beta - 1 \text{ and } p > 2\alpha + 1 \tag{3.6.64}$$

and if ϵ and $1/\gamma$ are sufficiently small.

Unfortunately, (3.6.63) has not been verified for $0 < \beta < 1$ in light of which we see that the assumption (3.6.62) does not rule out any β for which we can show that s_2 is less than or equal to zero.

In summary, therefore, we have considered that signs of both expressions S_1 and S_2 as described by (3.6.52) and (3.6.53) respectively. We first saw that S_1 is less than or equal to zero throughout $\Omega^+_{\delta_0} x (0,T)$ as required if ϵ and $1/\gamma$ are sufficiently small and if $\beta \geq 2$ and the conditions (3.6.56) and (3.6.61) are satisfied.

Conditions (3.6.56) and (3.6.61) require that

$$p > s > 1 \quad \text{and} \quad 2(p-\alpha)/\beta \geq s + 1 \tag{3.6.65}$$

and must be satisfied in addition to (3.5.4). Some $s > 1$ exists to satisfy (3.6.65) and for which S_1 is less than or equal to zero, therefore, provided

$$2(p-\alpha)/\beta \geq s + 1 > 2 \ ,$$

i.e.

$$p > \alpha + \beta \ , \tag{3.6.66}$$

and if $\beta \geq 2$ and $p > 1, \ \alpha \geq 0$ (from (3.5.4)).

We next considered S_2 in the case that $\beta < 2$. By making reference to previous work we were able to conclude that there exists some $s > 1$ for which S_2 is less than or equal to zero throughout $\Omega^+_{\delta_0} x (0,T)$. This conclusion again requires that ϵ and $1/\gamma$ are sufficiently small and in this case that the condition (3.6.64) is satisfied in addition to (3.5.4). Hence we require that

$2 > \beta \geq 1$ with

$$p > \alpha + \beta \quad \text{if} \quad \beta > 1 \text{ , or } \quad p > 2\alpha + 1 \quad \text{if} \quad \beta = 1 \tag{3.6.67}$$

and $p > 1,\ \alpha \geq 0$ (from (3.5.4)).

To continue, for those $\alpha,\ \beta$ and p for which we have been able to show that either s_1 or s_2 is less than or equal to zero, we may apply the maximum principle to the appropriate equation (either (3.6.44) or (3.6.48)) to see that a positive maximum of $J(x,t)$ as defined by (3.6.42) is impossible within the interior of $\Omega^{+}_{\delta_0} x (0,T)$.

From (3.6.42) and (3.6.51) we further observe that

$$\begin{aligned} J(x,t) &= u_{x_1}(x,t) + \epsilon(x_1-\delta_0)^{\gamma} u^{s}(x,t) \\ &= u_{x_1}(x,t) \\ &< 0 \\ &\qquad \text{for } \{x_1 = \delta_0\} \text{ by (3.6.41)} \end{aligned}$$

and as

$$J(x,0) = \varphi_{x_1}(x) + \epsilon(x_1-\delta_0)^{\gamma}\varphi^{s}(x)$$

with x_1 an outward pointing vector to $\partial\Omega$, the condition (3.6.30) will ensure that $J(x,0) < 0$ throughout $\Omega^{+}_{\delta_0}$ provided ϵ is sufficiently small.

Finally, if Γ is defined as

$$\Gamma = \Omega^{+}_{\delta_0} \cap \partial\Omega$$

then as $u = 0$ on $\partial\Omega$ and $\frac{\partial u}{\partial x_1} < 0$ for $x \in \Omega^{+}_{\delta_0}$

by (3.6.41) it follows that

$$\begin{aligned} J(x,t) &= u_{x_1}(x,t) + \epsilon(x_1-\delta_0)^{\gamma} u^{s}(x,t) \\ &= u_{x_1}(x,t) \\ &< 0 \end{aligned}$$

$$\text{for } \; x \in (\Omega^{+}_{\delta_0} \cap \partial\Omega - \Gamma) \times (0,T) \; .$$

Hence, a positive maximum of J is also impossible on the parabolic boundary of $\Omega^{+}_{\delta_0} \times (0,T)$. In those cases where we have shown that either S_1 or S_2 is less than or equal to zero within $\Omega^{+}_{\delta_0} \times (0,T)$ we conclude from the maximum principle that $J < 0$ within $\Omega^{+}_{\delta_0} \times (0,T)$, and hence that

$$-u_{x_1} = |u_{x_1}| > \epsilon(x_1-\delta_0)^{\gamma} u^{s} \qquad \text{in} \quad \Omega^{+}_{\delta_0} \times (0,T) \tag{3.6.68}$$

where $\delta_0 \leq x_1 < 0$.

As $u(x,t) > 0$ within $\Omega^{+}_{\delta_0}$, it follows on integrating (3.6.68) with respect to x_1, that for each $y_1 \in \Omega^{+}_{\delta_0}$, i.e. $\delta_0 < y_1 < 0$,

$$\frac{1}{(s-1)\,u^{s-1}(y_1,0,t)} - \frac{1}{(s-1)\,u^{s-1}(\delta_0,0,t)} > \frac{\epsilon(y_1-\delta_0)^{\gamma+1}}{\gamma+1}$$

and, as $u(\delta_0,0,t) > 0$ and $s > 1$ that

$$u(y_1,0,t) \le \{\,\epsilon(s-1)\,(y_1-\delta_0)^{\gamma+1}/(\gamma+1)\}^{-1/(s-1)}$$

for $\delta_0 < y_1 < 0,\ 0 < t < T$. (3.6.69)

Hence, if $\delta_1 - \delta_0 - \epsilon_0$ for any $\epsilon_0 > 0$ then from (3.6.69),

$$u(y_1,0,t) \le \{\,\epsilon(s-1)\,(\delta_1-\delta_0)^{\gamma+1}/(\gamma+1)\}^{-1/(s-1)}$$

for any $\delta_1 < y_1 < 0,\ 0 < t < T$. (3.6.70)

The inequality (3.6.70) allows the conclusion that every point in the set

$$\{\,x' - 0,\ \delta_1 - \delta_0 - \epsilon_0 < x_1 < 0\,\}$$

is not a blow-up point for small enough ϵ_0 .

It is also clear that δ_0 (and hence δ_1) may be chosen independently of the initial point $y_0 \in \partial\Omega$. By varying y_0 along $\partial\Omega$ we conclude that there exists a neighbourhood, Ω' , of $\partial\Omega$, such that each point x in Ω' is not a blow-up point. We may now state the following Theorem.

Theorem 3.6.3

Suppose $\varphi(x)$ is large enough compared to Ω and satisfies (3.5.5) and that α, β and p satisfy (3.5.4), then the solution, u, to problem (3.5.1)-(3.5.3) will, according to Theorem 3.5.1, blow up at some finite time T.

Suppose further that Ω is a convex domain in $\mathbf{R}^N$ and that $\varphi(x)$ also satisfies the condition (3.6.30), then the set of blow-up points is a compact subset of Ω provided α, β and p satisfy, in addition,

$$p > \alpha + \beta \quad \text{if} \quad \beta > 1 , \qquad \text{or} \quad p > 2\alpha + 1 \quad \text{if} \quad \beta = 1 .$$

Remark

The final condition appearing in the statement of Theorem 3.6.3 simply indicates those values of α, β and p for which we have been able to show that one or other of the expressions S_1 or S_2 is less than or equal to zero throughout $\Omega^+_{\delta_0} x (0, T)$.

Section 3.7 Estimate of blow-up rate for the positive gradient case

In this section we seek to establish estimates of the rate at which a solution to problem (3.5.1)-(3.5.3) may blow-up. This analysis is very similar to that of Section 3.4 and we again require that the considered function blows up within a compact subset of the domain Ω . Hence, throughout this section we consider a solution to problem (3.5.1)-(3.5.3) which satisfies the requirements of Theorem 3.6.3 so that blow-up occurs at some finite time T within a compact subset of the convex domain Ω .

We begin by observing that the arguments used in Section 2.4 to establish Theorem 2.4.1 are (as in Section 3.4) directly applicable in this case and along with Corollary 2.4.2 now yield

Theorem 3.7.1

If the function $m(t)$ is defined as

$$m(t) = \max_{x \in \Omega} u(x, t) \tag{3.7.1}$$

them m is Lipschitz continuous and

$$m'(t) \leq m^p(t) \tag{3.7.2}$$

at any point at which the function m is differentiable.

The proof of Theorem 3.7.1 is identical to that of Theorem 2.4.1 and is not repeated.

Hence, as $m(T) = +\infty$, we integrate (3.7.2) from t to T to see that

$$m(t) = \max_{x\in\Omega} u(x,t) \geq \frac{C}{(T-t)^{1/(p-1)}} \qquad \text{for any} \quad 0 < t < T \,, \tag{3.7.3}$$

and for some constant $C > 0$.

We proceed to establish a complimentary estimate to (3.7.3) and assume that the initial condition $\varphi(x)$ satisfies

$$\varphi \in C^2(\Omega) \qquad \text{with} \quad \nabla^2\varphi + \varphi^p + \varphi^\alpha |\nabla\varphi|^\beta \geq 0 \quad \text{in} \quad \Omega \,. \tag{3.7.4}$$

This ensures that $u_t(x,0) \geq 0$ in Ω , and, as $u_t = 0$ on $\partial\Omega$ for $0 < t < T$, that u_t is no less than zero on the parabolic boundary of $\Omega_T = \Omega x(0,T)$.

Next, if we take $w(x,t)$ to denote $u_t(x,t)$, then on differentiating (3.5.1) with respect to t we see that

$$w_t = \nabla^2 w + pu^{p-1}w + \alpha u^{\alpha-1} |\nabla u|^\beta w + \beta u^\alpha |\nabla u|^{\beta-2} \nabla u . \nabla w \quad \text{in} \quad \Omega_T \,. \tag{3.7.5}$$

Hence, as $u > 0$ throughout Ω_T and as $\beta \geq 1$ (from Theorem 3.6.3) it follows that each of the coefficients in equation (3.7.5) must remain bounded throughout the interior of Ω_T and that a negative minimum of w is impossible within the interior of Ω_T from the maximum principle.

We see, therefore, that

$$w - u_t > 0 \qquad \text{for} \quad x \in \Omega,\ 0 < t < T \tag{3.7.6}$$

and next derive a 'better' lower bound for u_t away from the parabolic boundary of Ω_T .

For any $\eta > 0$, define the set of Ω^η as

$$\Omega^\eta = \{ x \in \Omega : dist(x, \partial\Omega) > \eta \} \tag{3.7.7}$$

where for any point x and any set A

$$dist(x, A) = \min_{y \in A} \|x-y\| \ .$$

As in Section 3.4, we consider the function $J(x,t)$ defined as

$$J(x,t) = u_t(x,t) - \epsilon e^{-Mt} u^s(x,t) \tag{3.7.8}$$

for some small $\epsilon > 0$, positive M and $s > 1$.

For small enough ϵ , large enough M and suitable s we shall show that

$J > 0$ within $\Omega^\eta x(\eta, T)$.

We begin by observing that if η is small enough, then from Theorem 3.6.3 we have that

$$u^s \le c_0 < \infty \qquad \text{if} \quad x \in \partial\Omega^\eta,\ 0 < t < T \ . \tag{3.7.9}$$

Further, from (3.7.6) it follows that

$$u_t \geq c > 0 \quad \text{on the parabolic boundary of } \Omega^\eta x(\eta, T) \ . \tag{3.7.10}$$

From (3.7.9) and (3.7.10) we see that $J(x, t)$ can be forced to be strictly positive on the parabolic boundary of $\Omega^\eta x(\eta, T)$ if we choose ϵ small enough.

Next, on differentiating (3.7.8) it follows that

$$J_t - \nabla^2 J - (pu^{p-1} + \alpha u^{\alpha-1} |\nabla u|^\beta) J - \beta u^\alpha |\nabla u|^{\beta-2} \nabla u . \nabla J = S \tag{3.7.11}$$

where

$$S = \epsilon e^{-Mt} \{ (p-s) u^{p+s-1} + \alpha u^{\alpha+s-1} |\nabla u|^\beta \\ + (\beta-1) s u^{\alpha+s-1} |\nabla u|^\beta + M u^s + s(s-1) u^{s-2} |\nabla u|^2 \} . \tag{3.7.12}$$

As $u > 0$ throughout $\Omega^\eta x(\eta, T)$ and as $\beta \geq 1$ (from Theorem 3.6.3) the coefficients as the left hand side of equation (3.7.11) must remain bounded within $\Omega^\eta x(\eta, T)$. Further, S (as described by (3.7.12)) will be greater than or equal to zero throughout $\Omega^\eta x(\eta, T)$ if

$$p \geq s > 1 \ . \tag{3.7.13}$$

Hence as $J > 0$ on the parabolic boundary of $\Omega^\eta x(\eta, T)$ and as the maximum principle applied to equation (3.7.11) shows that J cannot have a negative minimum within $\Omega^\eta x(\eta, T)$ if s satisfies (3.7.13), it follows that

$$J - u_t - \epsilon e^{-Mt} u^s > 0 \quad \text{in} \quad \Omega^\eta x(\eta, T) \tag{3.7.14}$$

if both η and ϵ are small enough.

If (3.7.14) is integrated from t to T for any $\eta < t < T$ then, as $s > 1$ and $u > 0$ throughout $\Omega^\eta x(\eta, T)$,

$$u(x,t) \leq \frac{C}{(T-t)^{1/(s-1)}} \quad \text{for} \quad (x,t) \in \Omega^\eta (\eta, T)$$

and some positive constant C .

Further, by choosing η small enough it can also be established, in light of Theorem 3.6.3, that u must also remain bounded at any other point within Ω_T . Hence

$$u(x,t) \leq \frac{C_1}{(T-t)^{1/(s-1)}} \quad \text{for} \quad (x,t) \in \Omega_T \tag{3.7.15}$$

for some constant $C_1 > 0$ and s satisfying (3.7.13).

This yields the following Theorem.

Theorem 3.7.2

If $u(x,t)$ is a solution to problem (3.5.1)-(3.5.3) for which Theorem 3.6.3 applies, then as $s - p$ satisfies (3.7.13), there exists a constant $0 < C_1 < \infty$ such that

$$u(x,t) \le \frac{C_1}{(T-t)^{1/(p-1)}} \qquad \text{for} \quad (x,t) \in \Omega_T$$

and where T denotes the finite blow-up time of u .

Appendix A An Upper Bound for $|\nabla u|$

The purpose of this appendix is to derive an upper bound for the gradient of the solution to the type of problems considered in Chapters 2 and 3.

This analysis is, however, independent of the work of these chapters and the equations are therefore expressed in general terms.

We specifically consider those problems discussed in Chapters 2 and 3 where the gradient term is less than or equal to zero throughout the considered region.

Hence, $u(x,t)$ denotes the solution to the following problem

$$u_t = \nabla^2 u + u^p - u^\alpha|\nabla u|^\beta \quad \text{in} \quad \Omega, t > 0 \tag{A.1}$$

$$u(x,0) = \varphi(x) > 0 \quad \text{in} \quad \Omega, \tag{A.2}$$

$$u(x,t) = 0, \quad \text{on} \quad \Omega, \ t > 0 \tag{A.3}$$

where $p > 1$ and, minimally, $\alpha \geq 0, \beta \geq 1$ (these conditions may be superseded at some later point).

We also assume that

$$\Omega \text{ is a convex domain in } \mathbf{R}^N, \tag{A.4}$$

and that $\varphi(x)$ satisfies

$$\varphi \in C^1(\overline{\Omega}), \varphi \geq 0 \quad \text{and} \quad \varphi = 0 \quad \text{on} \quad \partial\Omega. \tag{A.5}$$

In the following we shall show that, if in addition,

$$p > \alpha + \beta \quad \text{and} \quad \beta > 2(p-\alpha)/(p+1),$$

then

$$|\nabla u|^2 \leq (Cu^m + u^k - Au^l + B)^2$$

for all $x \in \Omega, t > 0$ where $m - (p-\alpha)/\beta > k > l > 1$ and A, B and C are appropriate positive constants.

Before describing the proof of this result, we first make some observations which led to this choice of upper bound for $|\nabla u|$.

In equation (A.1), as the term $u^\alpha|\nabla u|^\beta$ is always greater than or equal to zero, we may estimate

$$u_t \leq \nabla^2 u + u^p \quad \text{in} \quad \Omega, t > 0. \tag{A.6}$$

From this point, if the symmetric case is considered as a particular example, i.e. if Ω is a ball, and initial data $\varphi(x)$ is assumed to satisfy the conditions necessary to ensure that

(i) u is radially symmetric with $u_r < 0$, and

(ii) $u_t > 0$ for all $x \in \Omega, t > 0$,

then on multiplying (A.6) by u_r we obtain that

$$0 \geq u_r u_{rr} + u^p u_r .$$

If this expression is now integrated with respect to r, we find that

$$\tfrac{1}{2}u_r^2(r) \le \int_{u(r)}^{u(0)} s^p ds. \tag{A.7}$$

We thus see that this particular solution to equation (A1) satisfies an estimate similar to that obtained in Friedman & Macleod 1985 for the solution to equation (A.1) without the gradient term.

The problem in this case, however, is that by discarding the gradient term, we must consequently lose information, and hence the estimate (A.7), even if it were shown to hold for a general solution to equation (A.1), may not be the 'best possible' in terms of representing the true nature of the gradient of the solution to this equation.

Furthermore, as this gradient term will always be of negative sign, it will have the effect of "focusing" the temperature u towards points where the gradient of u is zero, and hence result in a more 'peaked' solution.

We would expect this effect to be noticeable in any subsequent "good" upper bound for the gradient of u.

We therefore look for an alternative to (A.7) in this case, and begin by considering the function

$$J(x,t) = |\nabla u|^2 - g(u), \quad \text{in} \quad \Omega,\ t > 0, \tag{A.8}$$

where we look for some $g(u)$ to make $J<0$ in $\Omega, t>0$.

We shall assume, in addition, that $g(u) \ge 0$, for $x \in \Omega$, $t > 0$, with $g(0) > 0$ to give us a chance at satisfying $J<0$, on $\partial\Omega$.

On differentiating (A.8), we find that

$$J_t = 2\nabla u.\nabla u_t - g'(u)\, u_t, \tag{A.9}$$

$$J_{x_i} = 2\nabla u.\nabla u_{x_i} - g'(u)\, u_{x_i}, \tag{A.10}$$

$$\nabla^2 J = 2\sum_i |\nabla u_{x_i}|^2 + 2\nabla u.\nabla(\nabla^2 u) - g''(u)|\nabla u|^2 - g'(u)\nabla^2 u, \tag{A.11}$$

and hence

$$\begin{aligned} J_t - \nabla^2 J = 2pu^{(p-1)}|\nabla u|^2 - 2\alpha u^{\alpha-1}|\nabla u|^{\beta+2} \\ - \beta u^{\alpha}|\nabla u|^{\beta-2} 2\sum_i u_{x_i}(\nabla u.\nabla u_{x_i}) - g'(u)\, u^p \\ + g'(u)\, u^{\alpha}|\nabla u|^{\beta} - 2\sum_i |\nabla u_{x_i}|^2 + g''(u)|\nabla u|^2. \end{aligned} \tag{A.12}$$

Using (A.10) to substitute for $\nabla u.\nabla u_{x_i}$ in (A.12) we see that

$$\begin{aligned} J_t - \nabla^2 J = 2pu^{p-1}|\nabla u|^2 - 2\alpha u^{\alpha-1}|\nabla u|^{\beta+2} - (\beta-1)\, u^{\alpha}|\nabla u|^{\beta} g'(u) \\ - \beta u^{\alpha}|\nabla u|^{\beta-2}(\nabla J.\nabla u) - g'(u)\, u^p - 2\sum_i |\nabla u_{x_i}|^2 + g''(u)|\nabla u|^2. \end{aligned} \tag{A.13}$$

Also from (A10)

$$2\nabla u.\nabla u_{x_i} = J_{x_i} + g'(u)\, u_{x_i},$$

so that

$$4|\nabla u|^2|\nabla u_{x_i}|^2 \geq 4(\nabla u.\nabla u_{x_i})^2 = (J_{x_i} + g'(u)\, u_{x_i})^2$$

and then

$$4|\nabla u|^2 \sum_i |\nabla u_{x_i}|^2 \geq |\nabla J|^2 + 2g'(u)\nabla J.\nabla u + g'^2(u)\; |\; \nabla u\; |^2. \tag{A.14}$$

From (A.8), we see that J cannot take a positive maximum where $|\nabla u| = 0$, as $J = -g \le 0$, where $|\nabla u| = 0$. Hence, if (A.14) is used to estimate $\sum_i |\nabla u_{x_i}|^2$ in (A.13), we find that

$$J_t - \nabla^2 J + \{\beta u^{\alpha}|\nabla u|^{\beta-2} + g'(u)/|\nabla u|^2\}(\nabla J.\nabla u) + \tfrac{1}{2}|\nabla J|^2/|\nabla u|^2 \le S \tag{A.15}$$

where

$$\begin{aligned} S = 2pu^{p-1}|\nabla u|^2 - 2\alpha u^{\alpha-1}|\nabla u|^{\beta+2} - (\beta-1)u^{\alpha}|\nabla u|^{\beta}g'(u) \\ - g'(u)u^p - \tfrac{1}{2}g'^2(u) + g''(u)|\nabla u|^2 \end{aligned} \tag{A.16}$$

and each of the coefficients on the left hand side of inequality (A.15) are bounded at any positive maximum of J.

Further, if $u=0$, then S will be less than or equal to zero provided

$$p \ge 1, \quad \alpha \ge 0, \quad g'(0) = 0, \quad \text{and} \quad g''(0) \le 0, \tag{A.17}$$

and hence, a positive maximum of J also cannot occur where $u = 0$.

If we now substitute for $|\nabla u|$ in (A.16) by using equation (A.8), then

$$\begin{aligned} J_t - \nabla^2 J + \{\beta u^{\alpha}|\nabla u|^{\beta-2} + g'(u)/|\nabla u|^2\}(\nabla J.\nabla u) + \tfrac{1}{2}|\nabla J|^2/|\nabla u|^2 \\ + 2\alpha u^{\alpha-1}\{(J+g(u))^{(\beta+2)/2} - g^{(\beta+2)/2}(u)\} \\ + (\beta-1)u^{\alpha}g'(u)\{(J+g(u))^{\beta/2} - g^{\beta/2}(u)\} \\ - \{2pu^{(p-1)} + g''(u)\}J \le S_1, \end{aligned}$$

where

$$S_1 = 2pu^{(p-1)}g(u) - 2\alpha u^{\alpha-1}g^{(\beta+2)/2}(u) - (\beta-1)u^{\alpha}g^{\beta/2}(u)g'(u) \\ - g'(u)u^{p} - \tfrac{1}{2}g'^{2}(u) + g''(u)g(u), \tag{A.18}$$

and where all the relevant coefficients of J on the left hand side of inequality (A.18) are also bounded at any positive maximum of J.

We first consider $g(u) = u^m$ for some $m > 0$. S_1 then becomes

$$(2p-m)u^{p+m-1} - (2\alpha + m(\beta-1))u^{\alpha-1+m(\beta+2)/2} - \tfrac{1}{2}m^2u^{2m-2} + m(m-1)u^{2m-2},$$

whereby, if this expression can be shown to be less than or equal to zero for all u, then we may conclude that u cannot attain a positive maximum within $\Omega \times (0,T)$.

From this expression we see that, if $2\alpha + m(\beta-1) > 0$, then for large u the second term dominates if

$$2(p-\alpha)/\beta < m, \tag{A.19}$$

and either $\beta \geq 2$ or $m < 2(\alpha+1)/(2-\beta)$.

With (A.19), at least one of $\beta \geq 2$ or $m < 2(\alpha+1)/(2-\beta)$, will always be satisfied provided,

$$\beta > 2(p-\alpha)/(p+1).$$

The second term therefore dominates, for large u, provided,

$$\beta > 2(p-\alpha)/(p+1) \quad \text{and} \quad m > 2(p-\alpha)/\beta,$$

although this inequality will fail as u becomes small, i.e. as we approach the boundary of Ω.

This analysis does, however, lead to the conclusion that, for large u, $g(u)$ need only grow as fast as u^m, where $m > 2(p-\alpha)/\beta$, provided

$\beta > 2(p-\alpha)/(p+1)$.

From this point we find that $g(u) - (u^m - Au^1 + B)^2$, with $m > (p-\alpha)/\beta$, will satisfy the condition required to ensure that $S_1 \le 0$, for all $x \in \Omega, t > 0$, i.e. we find that by including the extra terms in u^1, and u^0 that we can also ensure that S_1 is less than or equal to zero when u is 'small'. The choice of constants A and B is then made to ensure that our 'large' and 'small' regions overlap.

At first this form of $g(u)$ would appear adequate. However, in particular aspects of the analysis of Chapters 2 and 3 we find that we would like the highest power of u in $g(u)$ to be equal to $2(p-\alpha)/\beta$, and we would also like the coefficient of this term to be as small as possible. We subsequently find that this can be achieved by including an extra term in u^k with

$$m = (p-\alpha)/\beta > k > l > 1 \; .$$

We now set

$$g(u) = (Cu^m + u^k - Au^l + B)^2 \tag{A.20}$$

where $m = (p-\alpha)/\beta > k > l > 1,$ and consider the decomposition of β into its integer and non-integer parts, say $\beta = n + \delta$ where n is the largest integer such that $n \le \beta$ and $0 \le \delta < 1.$

We propose that, if the following conditions are satisfied:-

(i) $\beta > \dfrac{2(p-\alpha)}{(p+1)},$

(ii) $C^\beta[n(\alpha+m(\beta-1)) + \alpha] > p,$ and

(iii) $C \ge 1,$

with k 'close enough' to $m,$ then the above choice of $g(u)$ ensures that $s,$ as defined by (A.16), remains less than or equal to zero for all positive values of $u.$

It is clear, even at this early stage, that the resulting expressions for s_1 and its derivatives will be complex. To try to alleviate this problem we shall henceforth denote

$$(Cu^m + u^k - Au^l + B) = g^{1/2}(u) \quad \text{by} \quad h(u) \quad \text{or just} \quad h.$$

On differentiating (A.20) with respect to u we see that

$$g'(u) = 2(Cu^m + u^k - Au^l + B)(Cmu^{m-1} + ku^{k-1} - Alu^{l-1}) \tag{A.21}$$

and

$$\begin{aligned} g''(u) = {} & 2h(Cm(m-1)u^{m-2} + k(k-1)u^{k-2} - Al(l-1)u^{l-2}) \\ & +2(Cmu^{m-1} + ku^{k-1} - Alu^{l-1})^2, \end{aligned} \tag{A.22}$$

so that $g'(0) = 0$ and $g''(0) \leq 0$ as required by (A.17).

On substituting these relations into (A.18) we find that if we choose $A, B > 0$ and $k > l > 1$ such that $u^k - Au^l + B > 0$ for all $u \in [0, \infty]$ then, if

$$S_2 = S_1/2h,$$

$$\begin{aligned} S_2 = {} & C(p-m)u^{p+m-1} + (p-k)u^{p+k-1} - A(p-l)u^{p+l-1} + pBu^{p-1} \\ & - u^{\alpha-1}h^{\beta}\{C(\alpha + m(\beta-1)u^m + (\alpha + k(\beta-1))u^k \\ & - A(\alpha + l(\beta-1))u^l + \alpha B\} \\ & + h^2\{Cm(m-1)u^{m-2} + k(k-1)u^{k-2} - Al(l-1)u^{l-2}\} \end{aligned} \tag{A.23}$$

and the sign of s_1 depends only on the sign of s_2.

As β may be non-integer valued, we consider $\beta - n + \delta$, where n is an integer, and $0 \leq \delta < 1$.

In the first stage of this analysis we show that s_2 is less than or equal to zero provided u is "large enough".

In order to do this, we shall write s_2 as the sum of terms defined by the highest integer power of B contained in their coefficients, i.e a term whose coefficient contains the term B^δ is considered as a term in B^0 etc.

We then examine each of these expressions in turn and try to show that, for large u, each individual term is itself less than zero so that s_2, which is the sum of these terms, must also be less than zero as required.

The terms in s_2 with B^0 as the highest integer power of B their coefficients are:-

$$\begin{aligned}
&C(p-m)\,u^{p+m-1} + (p-k)\,u^{p+k+1} - A(p-l)\,u^{p+l-1} \\
&\quad - u^{\alpha-1}\{Cu^m + u^k - Au^1\}^n h^\delta\{C(\alpha+m(\beta-1)\}u^m \\
&\qquad\qquad + (\alpha+k(\beta-1))\,u^k - A(\alpha+l(\beta-1))\,u^l\} \\
&\quad + \{Cu^m + u^k - Au^l\}^2\{Cm(m-1)\,u^{m-2} + k(k-1)\,u^{k-2} - Al(l-1)\,u^{l-2}\}
\end{aligned} \tag{A.24}$$

Noticing that $\{Cu^m + u^k - Au^l\} - h - B$ we adopt this notation in the subsequent expressions, and then B^1:

$$pu^{p-1} - u^{\alpha-1}h^{\delta}\{n(h-B)^{n-1}[C(\alpha+m(\beta-1))u^m + (\alpha+k(\beta-1))u^k - A(\alpha+l(\beta-1))u^l] + \binom{n}{0}(h-B)^n\alpha\} + 2(h-B)\{Cm(m-1)u^{m-2} + k(k-1)u^{k-2} - Al(l-1)u^{l-2}\} \tag{A.25}$$

B^2:

$$-u^{\alpha-1}h^{\delta}\{\tfrac{1}{2}n(n-1)(h-B)^{n-2}[C(\alpha+m(\beta-1)]u^m + (\alpha+k(\beta-1))u^k - A(\alpha+l(\beta-1))u^l] + n(h-B)^{n-1}\alpha\} + Cm(m-1)u^{m-2} + k(k-1)u^{k-2} - Al(l-1)u^{l-2} \tag{A.26}$$

if $\beta \geq 2,$ otherwise the term in $(h-B)^{n-2}$ disappears, and, for $2 < j \leq n,$

B^j:

$$-u^{\alpha-1}h^{\delta}\left\{\binom{n}{j}(h-B)^{n-j}[C(\alpha + m(\beta-1))u^m + (\alpha+k(\beta-1))u^k - A(\alpha+l(\beta-1))u^l] + \binom{n}{j-1}(h-B)^{n-(j-1)}\alpha\right\} \tag{A.27}$$

The highest integer power of B is B^{n+1} and has coefficient

$$-u^{\alpha-1}\alpha h^{\delta}. \tag{A.28}$$

We shall show that the coefficient of each integer power of B is of negative sign when u is greater than some constant which is independent of B.

Clearly the coefficient of B^{n+1} is less than or equal to zero as required.

We now consider the coefficients of those integer powers of B which are greater than 2, where we assume that u is large enough, compared to A, so that,

$$\tfrac{1}{2}u^{k} > Au^{l}. \tag{A.29}$$

On considering the term in B^{j}, for any $j > 2$, we see that (A.29) ensures that this is less than or equal to zero as required, as $k > l$.

We now consider the B^2 term, as given by (A.26), and propose that

$$\begin{aligned} Cm(m-1)u^{m-2} + k(k-1)u^{k-2} - \tfrac{1}{2}Al(l-1)u^{l-2} \\ \leq \tfrac{1}{2}u^{\alpha-1+m(\beta-1)}C^{\beta-1}\{\tfrac{1}{2}n(n-1)(\alpha + m(\beta-1)) + \alpha n\} \end{aligned} \tag{A.30}$$

for all $u \in [0,\infty]$ if A is large enough. We see that this will be true if either

$$Cm(m-1)u^{m-2} + k(k-1)u^{k-2} \leq \tfrac{1}{2}Al(l-1)u^{l-2}, \tag{A.31}$$

or if

$$\begin{aligned} Cm(m-1)u^{m-2} + k(k-1)u^{k-2} \\ \leq \tfrac{1}{2}u^{\alpha-1+m(\beta-1)}C^{\beta-1}\{\tfrac{1}{2}n(n-1)(\alpha + m(\beta-1)) + \alpha n\}. \end{aligned} \tag{A.32}$$

Inequality (A.31) is satisfied if u is small enough compared to A (this follows as $m > k > l$). Inequality (A.32) is satisfied if

$$\alpha - 1 + m(\beta-1) > m-2 > k-2, \quad \text{i.e. if} \quad \beta > 2(p-\alpha)/(p+1), \tag{A.33}$$

as $m - (p-\alpha)/\beta,$ and if u is greater than some constant which is independent of A and proportional to some negative power of C.

Hence, if A is large enough, one of these will be true for all $u \in [0,\infty]$. With this we find that the B^2 term is less than

$$\begin{aligned} - u^{\alpha-1}h^{\delta}\{\tfrac{1}{2}n(n-1)\,(h-B)^{n-2}[C(\alpha+m(\beta-1))\,u^{m} \\ + (\alpha+k(\beta-1))\,u^{k} - A(\alpha+l(\beta-1)\,u^{l}] + n(h-B)^{n-1}\alpha\} \\ + \tfrac{1}{2}u^{\alpha-1+m(\beta-1)}C^{\beta-1}\{\tfrac{1}{2}n(n-1)\,(\alpha+m(\beta-1))+\alpha n\}. \end{aligned} \tag{A.34}$$

If we maintain assumption (A.29), we may estimate

$$h > (h-B) > Cu^{m},\ (\alpha+k(\beta-1))\,u^{k} > A(\alpha + l(\beta-1))\,u^{l}, \tag{A.35}$$

so that expression (A.34) is now less than

$$\begin{aligned} -u^{\alpha-1}(Cu^{m})^{\delta}\{\tfrac{1}{2}n(n-1)\,(Cu^{m})^{n-2}C(\alpha+m(\beta-1))\,u^{m} + n(Cu^{m})^{n-1}\alpha\} \\ + \tfrac{1}{2}u^{\alpha-1+m(\beta-1)}C^{\beta-1}\{\tfrac{1}{2}n(n-1)\,(\alpha+m(\beta-1)) + \alpha n\}. \end{aligned}$$

As $\beta - n + \delta,$ this expression reduces to

$$- \tfrac{1}{2}u^{\alpha-1+m(\beta - 1)}C^{\beta-1}\{\tfrac{1}{2}n(n-1)\,(\alpha + m(\beta - 1)) + \alpha n\}$$

and is less than or equal to zero as required.

We next consider the term in B^1, as given by (A.25), and again use the estimates (A.35), so that this expression is less than

$$- u^{\alpha-1}(Cu^{m})^{\delta}\{n(Cu^{m})^{n-1}(C(\alpha + m(\beta - 1))u^{m} + \alpha(Cu^{m})^{n}\}$$
$$+ 2(h-B)\{Cm(m-1)u^{m-2} + k(k-1)u^{k-2} - Al(l-1)u^{l-2}\} + pu^{p-1}.$$

As $(h - B) \geq 0$, and $m - (p - \alpha)/\beta$, this expression is in turn less than

$$u^{p-1}\{p - C^{\beta}[n(\alpha + m(\beta - 1)) + \alpha]\} + 2(Cu^{m} + u^{k})\{Cm(m-1)u^{m-2} + k(k-1)u^{k-2}\}. \tag{A.36}$$

If we now take C large enough so that

$$C^{\beta}\{n(\alpha + m(\beta - 1)) + \alpha\} > p, \tag{A.37}$$

then the term in u^{p-1} is of negative sign, and then if

$$p - 1 > 2m - 2 > 2k - 2, \quad \text{i.e.} \quad \beta > 2(p - \alpha)/(p + 1), \tag{A.38}$$

condition (A.29) ensures that (A.36), and hence the term in B^1, is also negative as required, if u is large enough.

Finally, we consider the term in B^0 as given by (A.24). We shall be maintaining assumption (A.29), but find that merely estimating $(h - B) > Cu^{m}$ is not sufficient in this case. We therefore split (A.24) into positive and negative terms and examine it more thoroughly. First we consider $(h - B)^n$ and see that this can be expressed as

$$\sum_{i\ even}\binom{n}{i}\{Cu^m + u^k\}^{n-i}(Au^l)^i - \sum_{i\ odd}\binom{n}{i}\{Cu^m + u^k\}^{n-i}(Au^l)^i. \tag{A.39}$$

But then

$$\begin{aligned}\sum_{i\ even}\binom{n}{i}\{Cu^m + u^k\}^{n-i}(Au^l)^i &\geq \{Cu^m + u^k\}^n \\ &\geq (Cu^m)^n + n(Cu^m)^{n-1}u^k,\end{aligned} \tag{A.40}$$

so that (A.24) will be less than or equal to zero if

$$\begin{aligned}&A(\alpha + l(\beta - 1))u^{\alpha+l-1}(h-B)^n h^\delta + C(p - m)u^{p+m-1} + (p - k)u^{p+k-1} - A(p-l)u^{p+l-1} \\ &\quad + (h-B)^2\{Cm(m - 1)u^{m-2} + k(k - 1)u^{k-2} - Al(l - 1)u^{l-2}\} \\ &\quad + u^{\alpha-1}h^\delta\left\{\sum_{i\ odd}\binom{n}{i}\{Cu^m + u^k\}^{n-i}(Au^l)^i\right\}\{C(\alpha + m(\beta - 1))u^m \\ &\qquad\qquad + (\alpha + k(\beta - 1)u^k\} \\ &\leq u^{\alpha-1}h^\delta\{C(\alpha + m(\beta - 1)u^m + (\alpha + k(\beta - 1)u^k\}\{(Cu^m)^n + (Cu^m)^{n-1}u^k\}.\end{aligned} \tag{A.41}$$

We again estimate $h \geq Cu^m$ and, as $n + \delta = \beta$, the right hand side of (A.41) is greater than or equal to

$$\begin{aligned}&(\alpha + m(\beta-1))C^{\beta+1}u^{p+m-1} + C^{n-1}n(\alpha + k(\beta-1))u^{m(n-1)+2k+\alpha-1}h^\delta \\ &\qquad \{(\alpha + k(\beta-1)) + n(\alpha+m(\beta-1))\}C^\beta u^{p+k-1},\end{aligned}$$

where we have used $m = (p-\alpha)/\beta$.

Inequality (A.41) is therefore satisfied if

$$(\alpha + m(\beta-1))C^{\beta+1}u^{p+m-1} + \{(\alpha+k(\beta-1)) + n(\alpha + m(\beta-1))\}C^{\beta}u^{p+k-1} \geq$$
$$C(p-m)u^{p+m-1} + (p-k)u^{p+k-1} + (h-B)^2\{Cm(m-1)u^{m-2} + k(k-1)u^{k-2}\}, \tag{A.42}$$

which we call condition A and

$$h^{\delta}C^{n-1}n(\alpha + k(\beta-1))u^{m(n-1)+2k+\alpha-1} \geq A(\alpha + l(\beta-1))u^{\alpha+l-1}(h-B)^n h^{\delta}$$
$$+ u^{\alpha-1}h^{\delta}\left\{\sum_{i\ odd}(Cu^m+u^k)^{n-i}(Au^l)^i\right\}\{C(\alpha+m(\beta-1))u^m + (\alpha+k(\beta-1))u^k\}, \tag{A.43}$$

which we call condition B.

Condition A will be satisfied, for u large enough, if either

(i) $(\alpha + m(\beta - 1))C^{\beta} > (p-m)$, i.e. $C > 1$, and $p + m - 1$ is the highest power of u in (A.42)

or (ii) $C - 1$, so that the terms in u^{p+m-1} cancel, and then $(p + k - 1)$ is the highest remaining power of u in (A.42) and is of the desired sign, which it will be if

$$\{\alpha + k(\beta-1) + n(\alpha + m(\beta-1))\} > (p-k).$$

On simplifying the above two conditions we conclude that condition A will be satisfied, for u large enough compared to A, provided either,

(i) $C > 1$ and $(p + m - 1) > 3m - 2,$ (A.44) or, as $m > k$, (ii) $C - 1, \{\alpha + k(\beta-1) + n(\alpha + m(\beta-1))\} > (p-k)$, and

$$p + k - 1 > 3m - 2, \tag{A.45}$$

where n, is the largest integer satisfying $n \leq \beta$.

As $m - (p-\alpha)/\beta$ the inequality $p + m - 1 > 3m - 2$ is satisfied if

$\beta > 2(p-\alpha)/(p+1)$, and $p + k - 1 > 3m - 2$ requires $\beta > 3(p-\alpha)/(p+k+1)$.

Condition *B* will be satisfied, for u large enough compared to A, provided

$$m(n-1) + 2k + \alpha - 1$$

is the highest power of u in equality (A.43), which it will be if

$$2k > m + 1. \tag{A.46}$$

We conclude, therefore, that if condition (A.46) holds, and at least one of either (A.44) or (A.45) is also satisfied, then for u large enough compared to A the conditions *A* and *B* will be true, and hence the term in B^0 is of the desired sign.

If we now combine all the requirements for each integer power of B to be less than or equal to zero, we find that, if the following four conditions are satisfied,

(i) $\beta > 2(p-\alpha)/(p+1)$, (A.33)

(ii) $C^\beta[n(\alpha + m(\beta-1)) + \alpha] > p$, (A.37)

(iii) either $C > 1$, and $\beta > 2(p-\alpha)/p+1)$, (A.44)

(iii) either $C > 1$, and $\beta > 2(p-\alpha)/p+1)$, (A.44)

or $C = 1$, $\{\alpha + k(\beta-1) + n(\alpha + m(\beta-1))\} > (p-k)$ and

$\beta > 3(p-\alpha)/(p+k+1)$, (A.45)

(iv) $2k > m + 1$, (A.46)

and if u is large enough compared to A, then S, as defined by (A.16), as it is the sum of these terms, is also less than or equal to zero as required.

We now try to show that S is less than or equal to zero when u is 'small'.

We recall that inequality (A.30) is independent of the assumption that u is large, and so by using it again to substitute for the term

$$h^2\{Cm(m-1)u^{m-2} + k(k-1)u^{k-2}\}$$

in (A.23), we find that

$$\begin{aligned}
S_2 \le\ & C(p-m)u^{p+m-1} + (p-k)u^{p+k-1} - A(p-l)u^{p+l-1} + pBu^{p-1} \\
& - u^{\alpha-1}h^{n+\delta}\{C(\alpha + m(\beta-1))u^m \\
& \quad + (\alpha + k(\beta - 1)u^k - A(\alpha + l(\beta-1))u^l + \alpha B\} \\
& \quad\quad + (h^2-B^2)\{Cm(m-1)u^{m-2} + k(k-1)u^{k-2} - Al(l-1)u^{l-2}\} \\
& \quad\quad\quad + \tfrac{1}{2}B^2u^{\alpha-1+m(\beta-1)}C^{\beta-1}\{\tfrac{1}{2}n(n-1)(\alpha + m(\beta-1)) + \alpha n\} \\
& \quad\quad\quad\quad - \tfrac{1}{2}B^2Al(l-1)u^{l-2}.
\end{aligned}$$

We again wish to show that $S_2 \le 0$, and see that this will be satisfied if,

$$C(p-m)u^{p+m-1} + (p-k)u^{p+k-1} + pBu^{p-1} + (h^2-B^2)\{Cm(m-1)u^{m-2} + k(k-1)u^{k-2} - Al(l-1)u^{l-2}\} \le \tfrac{1}{2}B^2Al(l-1)u^{l-2}, \tag{A.47}$$

which we call condition C, and

$$\tfrac{1}{2}B^2u^{(\alpha-1)+m(\beta-1)}C^{\beta-1}\{\tfrac{1}{2}n(n-1)(\alpha+m(\beta-1)) + \alpha n\} \le u^{\alpha-1}h^{n+\delta}\{C(\alpha + m(\beta-1))u^m - A(\alpha + l(\beta-1))u^l + \alpha B\}, \tag{A.48}$$

which we will call condition D.

Condition C will be satisfied if $(l-2)$ is the smallest power of u in inequality (A.47), and if u is small enough compared to B.

As $(h^2 - B^2)$ has u^l as its lowest power of u, the only other possibility for the smallest power of u in (A.47) is $2l-2$.

Hence $l > 0$, and u small enough compared to B ensures that condition C is satisfied as required. Condition D is equivalent to

$$\tfrac{1}{2}B^2u^{m(\beta-1)}C^{\beta-1}\{\tfrac{1}{2}n(n-1)(\alpha+m(\beta-1))+\alpha n\} + A(\alpha+l(\beta-1))u^lh^{n+\delta} \le h^{n+\delta}\{\alpha B + C(\alpha+m(\beta-1))u^m\}. \tag{A.49}$$

We now consider h^n;

$$h^n = \sum_{i\ even}\{Cu^m + u^k + B\}^{n-i}(Au^l)^i - \sum_{i\ odd}\{Cu^m + u^k + B\}^{n-i}(Au^l)^i,$$

and estimate

$$\sum_{i\ even} \{Cu^m + u^k + B\}^{n-i}(Au^l)^i \geq \{Cu^m + u^k + B\}^n$$
$$\geq \tfrac{1}{2}n(n-1)(Cu^m)^{n-2}B^2 + n(Cu^m)^{n-1}B + B^n.$$

On substituting these estimates into inequality (A.49), we see that condition D is satisfied if

$$\tfrac{1}{2}B^2u^{\alpha-1+m(\beta-1)}C^{\beta-1}\{\tfrac{1}{2}n(n-1)(\alpha+m(\beta-1)) + \alpha n\} + A(\alpha+1(\beta-1)u^{\alpha+l-1}h^{n+\delta}$$
$$+ u^{\alpha-1}h^{\delta}\{\alpha B + C(\alpha+m(\beta-1))u^m\}\sum_{i\ odd}\{Cu^m + u^k + B\}^{n-i}(Au^l)^i$$
$$\leq u^{\alpha-1}h^{\delta}\{\alpha B + C(\alpha+m(\beta-1))u^m\}\{\tfrac{1}{2}n(n-1)(Cu^m)^{n-2}B^2 + n(Cu^m)^{n-1}B + B^n\} \tag{A.50}$$

If B is large enough, then $u^k - Au^l + B > 0$ for all $u \in [0,\infty]$, so that

$h > Cu^m$, and the right hand side of (A.50) is greater than or equal to

$$u^{\alpha-1+m(\beta-1)}C^{\beta-1}\{\tfrac{1}{2}n(n-1)(\alpha+m(\beta-1)) + \alpha n\}B^2 + \alpha B^{n+1}h^{\delta}u^{\alpha-1}.$$

Condition D is therefore satisfied if

$$A(\alpha + l(\beta-1))h^{n+\delta}u^{\alpha+l-1}$$
$$+ u^{\alpha-1}h^{\delta}\{\alpha B + C(\alpha + m(\beta-1))u^m\}\sum_{i\ odd}\{Cu^m+u^k+B\}^{n-i}(Au^l)^i$$
$$\leq \alpha B^{n+1}h^{\delta}u^{\alpha-1},$$

which, on cancelling $u^{\alpha-1}h^{\delta}$ from both sides, reduces to

$$\{\alpha B + C(\alpha + m(\beta-1))u^m\}\sum_{i\ odd}\{Cu^m + u^k + B\}^{n-i}(Au^l)^i + A(\alpha + l(\beta-1))u^l h^n$$
$$\leq \alpha B^{n+1}. \tag{A.51}$$

We thus conclude that inequality (A.51) will be satisfied if u is small enough compared to B.

If we compare this with our previous conclusion, that S will be less than or equal to zero if u is large enough compared to A, and as we are free to choose the relative sizes of constants A and B, B chosen large enough compared to A ensures that S is less than or equal to zero for all positive values of u.

We may therefore conclude, from the maximum principle, that a positive maximum of J must be attained on the parabolic boundary of $\Omega\times(0,T)$.

Initially, however,

$$\begin{aligned} J(x,0) &= |\nabla\varphi(x)|^2 - g(\varphi(x)), \\ &= |\nabla\varphi(x)|^2 - \{C\varphi^m + \varphi^k - A\varphi^l + B\}^2, \end{aligned}$$

and, as $\varphi\in C^1(\Omega)$, $|\nabla\varphi(x)|$ is necessarily finite for each $x\in\Omega$, it follows that B large enough will ensure that $J(x,0) \le 0$ for each $x\in\Omega$.

Hence, a positive maximum of J cannot occur at $t = 0$.

Next, if (y,s) is any point on $\partial\Omega, t>0$, and we denote by ν the outward normal to $\partial\Omega$ at y, then as

$u =$ constant $= 0$, on $\partial\Omega$, (for $t = s$), it follows that

$$\nabla^2 u = u_{vv} + (N-1)\kappa u_v, \quad \text{at} \quad (y,s),$$

where N = number of dimensions and κ = the non-negative mean curvature of $\partial\Omega$ at y.

Hence

$$J_v = 2u_v u_{vv} - g'(u)u_v,$$

and, as $m > k > l > 1$ and $u = 0$ on $\partial\Omega$, it follows that $g'(u) = g(0) = 0$ on $\partial\Omega$,

$$\{g'(u) = 2(Cu^m + u^k - Au^l + B)(Cu^{m-1} + ku^{k-1} - Alu^{l-1}\},$$

so that

$$J_v = u_v\{\nabla^2 u - (N-1)\kappa u_v\}.$$

If we recall equation (A.1), we see that $\nabla^2 u = 0$ on $\partial\Omega$ with the result that

$$J_v = -(N-1)\kappa u_v^2 \leq 0 \quad \text{at} \quad (y,s),$$

which, from the strong maximum principle for parabolic equations, contradicts J having a positive maximum at any point on $\partial\Omega$.

We therefore conclude that

$$J = |\nabla u|^2 - g(u) \leq 0, \quad \text{in} \quad \Omega, t > 0$$

where

$$g(u) = (Cu^m + u^k - Au^l + B)^2, \qquad m = (p-\alpha)/\beta > k > l > 1,$$

and A, B and C are some positive constants with A large enough, B large compared to A, and provided the following four conditions are satisfied:-

(i) $\beta > 2(p-\alpha)/(p+1)$,

(ii) $C^\beta[n(\alpha + m(\beta-1)) + \alpha] > p$,

(iii) either $C > 1$, and condition (i), or

$C = 1$, $\{\alpha + k(\beta-1) + n(\alpha + m(\beta-1))\} > p-k$ and

$\beta > 3(p-\alpha)/(p+k+1)$,

and (iv) $2k > m + l$,

where n is the largest integer satisfying $n \leq \beta$.

If conditions (i)-(iv) above are examined closely, we see that they can be simplified as follows:-

Condition (iii) requires, either $C > 1$ and condition (i), or $C = 1$,

$$\alpha + k(\beta-1) + n(\alpha + m(\beta-1)) > p-k, \tag{A.52}$$

and

$$\beta > 3(p-\alpha)/(p+k+1). \tag{A.53}$$

However, if $c - 1$, then condition (ii) reduces to

$$n(\alpha + m(\beta-1)) + \alpha > p,$$

which automatically gives (A.52), as $k\beta > 0$.

The second requirement if $c - 1$ is that

$$\beta > 3(p-\alpha)/(p+k+1), \text{ i.e. } k > 3(p-\alpha)/\beta - (p+1), \tag{A.53}$$

As our only other condition on how large k can be is that $m > k$, we see that

$$m - (p-\alpha)/\beta > k > 3(p-\alpha)/\beta - (p+1), \text{ i.e. } \beta > 2(p-\alpha)/(p+1),$$

which is already required as condition (i).

From the above analysis we see that, if k is chosen 'close enough' to m, then conditions (i) and (ii) make condition (iii) automatic for any $c \geq 1$. Finally, condition (iv),

$$2k > m + 1,$$

is obviously satisfied with $m > k > 1$, which is a requirement already, and if k is close to m.

Conditions (i)-(iv) above can therefore be replaced by the following

(i) $\beta > 2(p-\alpha)/(p+1)$,

(ii) $C^{\beta}\{\alpha + n(\alpha + m(\beta-1))\} > p,$

(iii) $C \geq 1,$ and $m - (p-\alpha)/\beta > k > l > 1$

with k 'close enough' to m and n the largest integer satisfying $n \leq \beta$.

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